RELATIONS BETWEEN HYPERSURFACE CROSS RATIOS, AND A COMBINATORIAL FORMULA FOR PARTITIONS OF A POLYGON, FOR PERMANENT PREPONDERANCE, AND FOR NON-ASSOCIATIVE PRODUCTS

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This note improves, in two respects, the results of §3.6 of my paper *The hypersurface cross ratio.*[1] There it is shown that the number c_n of independent hypersurface cross ratios that can be formed of $2n$ forms in n variables is 2 for $n=2$, 5 for $n=3$, and 14 for $n=4$. The proof employs the relations between cross ratios obtained by some simple permutations of the forms; let R be the set of these relations. It is remarked that the cross ratios of $2n-1$ forms in n variables, and of $2n-1$ forms in $n-1$ variables, are connected by the same relations as the cross ratios of $2n$ forms in n variables, as far as these are consequences of the relations R, a "perhaps void restriction." We now prove that $c_n = C_{2n,n}/(n+1)$, and that the restriction is in fact void, so that a complete knowledge of the relations between the cross ratios of $2n-1$ forms, of $2n$ forms, and of $2n+1$ forms in n variables is obtained.[2] The corresponding theorems for generalized intersections and one more variable are established at the same time.

The same facts hold for a general class of function ratios, which includes hypersurface cross ratios and generalized intersections as very special cases. The number c_n of independent function ratios has a simple combinatorial meaning, and appears also as the number of partitions of a polygon by non-intersecting diagonals into triangles, or of a cyclically arranged set into non-interlaced subsets, as the number of possibilities of never losing majority (in an election or a game[3]), and as the number of different products of given terms in a given order, in a non-associative multiplication. For the combinatorial formula, seven proofs are given, six extended to generalizations.[4]

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[1] Bull. Amer. Math. Soc. vol. 51 (1945) pp. 976–984.

[2] For forms of a sufficiently high degree. Cf., on the other hand, for 5, 5 and 6 linear forms in 2, 3 and 3 variables respectively, §§3, 4, 5 of *The pentagon in the projective plane, with a comment on Napier's rule*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 985–989.

[3] Or for drops falling on a board one-half of which is supported, and similar physical schemes.

[4] For an eighth proof cf. P. Erdös and I. Kaplansky, *Sequences of plus and minus*, Scripta Mathematica vol. 12 (1946) pp. 73–75 (for $[f(n, n)]^2$, read $f(n, n)f(n+1, n+1)$, or permit only diagonal moves; in (4), read $m \leqq n$). I have made use of oral remarks by A. Dvoretzky (in 2.3–2.5) and E. Jabotinsky (in 1.1).