ON A LINKAGE THEOREM BY L. CESARI

SAMUEL EILENBERG

In connection with his work on Lebesgue area of surfaces, Lamberto Cesari (*Su di un problema di analysis situs dello spazio ordinario*, R. Istituto Lombardo di Scienze e Lettere, Classe di Scienze Matematiche e Naturali, Rendiconti (3) vol. 6 (1942) pp. 267–291) has stated and proved a linkage theorem (see below), which reveals a rather interesting property of Euclidean 3-space. In view of the simplicity of the theorem and the laboriousness of Cesari's proof, the following concise proof may be of some interest.

In Euclidean three space E^3 consider the set M consisting of the three axes X, Y, Z. Let δ be a positive number and let N be the set consisting of the four lines

$$X_\delta:\quad (y = 0, z = -\delta), \qquad Y_\delta:\quad (x = 0, z = \delta),$$
$$Z_\delta':\quad (x = \delta, y = \delta), \qquad Z_\delta'':\quad (x = -\delta, y = -\delta).$$

THEOREM OF CESARI. *Any closed path in E^3-M that has a distance greater than δ from M and is contractible in E^3-N also is contractible in E^3-M.*

Let $\mathcal{A}$ be the fundamental group of E^3-M and $\mathcal{B}$ the fundamental group of E^3-N. We shall assume that the same point of E^3 is used as base point in the definition of both $\mathcal{A}$ and $\mathcal{B}$ and that this point has a distance greater than δ from M.

Given any element $a\in\mathcal{A}$ select a closed path in E^3-M in the class a with distance greater than δ from M. Such a path lies also in E^3-N and determines an element $\phi(a)$ of $\mathcal{B}$. It is easy to see that ϕ is single-valued and yields a homomorphism $\phi:\mathcal{A}\rightarrow\mathcal{B}$.

Cesari's theorem can now be reformulated as follows:

THEOREM. *ϕ maps $\mathcal{A}$ isomorphically into a subgroup of $\mathcal{B}$.*

To prove the theorem we draw projections of M and N and establish generators and relations for $\mathcal{A}$ and $\mathcal{B}$. In terms of these generators the homomorphism ϕ is given an explicit form. The problem thus translates into a problem on free groups which is solved algebraically.

In looking at the diagrams one should consider the eye as the base point of the fundamental group. To each line of the diagram corresponds then an element of the group represented by an arrow which corresponds to the path leading rectilinearly from the eye to the be-

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