ON A CONJECTURE OF CARMICHAEL

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Carmichael [1][2] conjectured that for no integer n can the equation $\phi(x)=n$ (ϕ being Euler's totient) have exactly one solution. To support the conjecture, he showed that each n for which there is a unique solution must satisfy a restriction which implies $n>10^{37}$. In this note we prove the validity of restrictions considerably stronger than those of Carmichael, and raise the lower bound on n to 10^{400}.

We shall denote by X the set of all integers x for which $\phi(y)=\phi(x)$ implies $y=x$. (If the conjecture is correct, X is empty, and the theorems stated are vacuously satisfied.)

(1) THEOREM. *Suppose that $\bar{x}=\prod_A p_i^{a_i}$ is in X, where the p_i's are distinct primes and A is the range of the index i. Let $m=\prod_B p_i^{a_i-1}(p_i-1)\cdot\prod_C p_i^{c_i}$ where B and C are disjoint subsets of A (one of them may be empty) and $c_i\leqq a_i-1$ for i in C. Then if p is prime and $p-1=m$, we have $p\mid\bar{x}$.*

For if $p\nmid\bar{x}$, we have $\phi(p\cdot\prod_{A-B-C}p_i^{a_i}\cdot\prod_C p_i^{a_i-c_i})=\phi(\bar{x})$, contrary to the definition of X.

(1.1) COROLLARY. *Suppose, under the hypotheses of* (1), *that B has the following property: if q is prime and $q\mid(p_j-1)$ for some j in B, then $q\mid\bar{x}$. We must then have $p^2\mid\bar{x}$.*

For under this condition we have $p-1=\prod_D p_i^{d_i}$, D being a subset of A. So if $p\mid\bar{x}$ but $p^2\nmid\bar{x}$, then $\phi(\prod_{A-D}p_i^{a_i}\cdot\prod_D p_i^{a_i+d_i}/p)=\phi(\bar{x})$, contrary to the definition of X.

(1.2) COROLLARY. *If, in the hypotheses of* (1), *B is empty, we have $p^2\mid\bar{x}$.*

(1.3) COROLLARY. $4\mid\bar{x}$. *If f is a Fermat prime such that $f\mid\bar{x}$, then $f^2\mid\bar{x}$.*

(1.2) and (1.3) are Carmichael's original conditions. From (1.1) and (1.3) it follows that $\bar{x}$ is divisible by 3^2, 7^2, 43^2, 3^3 or $13^2,\cdots$. (By extending this list Carmichael showed both $\bar{x}$ and $\phi(\bar{x})$ to be greater than 10^{37}.)

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[2] Numbers in brackets refer to the references at the end of the paper.