A NOTE ON THE DERIVATIVES OF INTEGRAL FUNCTIONS

S. M. SHAH

1. **Introduction.** Let $f(z)=\sum_0^\infty a_n z^n$ be an integral function of order ρ and lower order λ, and $M(r)=\max_{|z|=r}|f(z)|$; $M'(r)=\max_{|z|=r}|f'(z)|$. In a recent paper [1][1] I have proved the following two theorems.

THEOREM A. *If $f(z)$ be any integral function of order ρ then*[2]

$$\limsup_{r\to\infty}\frac{\log\{rM'(r)/M(r)\}}{\log r}=\rho. \tag{1.1}$$

THEOREM B. *If $f(z)=\sum a_n z^n$ be an integral function of lower order λ and $a_n\geqq 0$ then*

$$\liminf_{r\to\infty}\frac{\log\{rM'(r)/M(r)\}}{\log r}=\lambda.$$

The condition that the coefficients a_n be real and non-negative is unnecessary. The purpose of this note is to prove the following two theorems and to deduce a number of interesting results.

THEOREM 1. *If $f(z)$ be an integral function of lower order λ $(0\leqq\lambda\leqq\infty)$ then*

$$\liminf_{r\to\infty}\frac{\log\{rM'(r)/M(r)\}}{\log r}=\lambda. \tag{1.2}$$

THEOREM 2. *For any integral function $f(z)$ we have*

$$\begin{aligned}\liminf_{r\to\infty} M'(r)/M(r) &\leqq \liminf_{r\to\infty}\nu(r)/r \leqq \limsup_{r\to\infty}\nu(r)/r\\ &\leqq \limsup_{r\to\infty} M'(r)/M(r),\end{aligned} \tag{1.3}$$

$$\begin{aligned}\liminf_{r\to\infty} M^{(s+1)}(r)/M^{(s)}(r) &\leqq \liminf_{r\to\infty}\nu(r)/r \leqq \limsup_{r\to\infty}\nu(r)/r\\ &\leqq \limsup_{r\to\infty} M^{(s+1)}(r)/M^{(s)}(r) \qquad (s=1,2,3,\cdots),\end{aligned} \tag{1.4}$$

where $f^{(s)}(z)$ is the sth derivative of $f(z)$, $M^{(s)}(r)=\max_{|z|=r}|f^{(s)}(z)|$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] A glance at the proof [1, pp. 1–2] shows that the result (1.1) holds when ρ is infinite. An alternative proof of Theorem A is to employ Lemma 4 and relation (8) of my paper [1, p. 1].