ALGEBRAS AND THEIR SUBALGEBRAS

A. H. DIAMOND AND J. C. C. McKINSEY

There exist sets of postulates for Boolean algebras which consist entirely of universal sentences[1] in three variables. From this the following statement clearly follows: If an algebra has the property that every subalgebra generated by three elements is a Boolean algebra, then the algebra is itself a Boolean algebra.

The question arises, whether the above statement can be strengthened to read: If an algebra has the property that every subalgebra generated by two elements is a Boolean algebra, then the algebra is itself a Boolean algebra. This question can be answered in the negative by the following theorem:

THEOREM 1. *There exists an algebra* Γ *which is not a Boolean algebra, but such that every subalgebra of* Γ *with two generators is a Boolean algebra.*

PROOF. If x and y are any entities, then by "$\{x, y\}$" we shall mean the set whose only members are x and y; and by "$\langle x, y\rangle$" we shall mean the ordered couple whose first member is x and whose second member is y.

Let H be the intersection of all sets K which satisfy the following conditions:

(i) $2\in K$, $3\in K$, and $4\in K$;

(ii) If x and y are any distinct elements of K, neither of which belongs to the other, then $\{x, y\}\in K$.

Let H_1 be the class which contains all the members of H, and in addition the number 5. (Thus H_1 has just one more member than has H—namely 5.)

Let G be the class of all ordered couples $\langle\alpha, \beta\rangle$ where α is any member of H_1, and β is either 0 or 1.

We wish now to define a unary operation $-$, and two binary operations $+$ and $\cdot$, over the set G, in such a way that the system $\Gamma=\langle G, +, \cdot, -\rangle$ will not be a Boolean algebra, but every subalgebra of Γ with two generators will be a Boolean algebra.

If $\langle\alpha, \beta\rangle$ is any member of G, then we set

Presented to the Society, December 28, 1946; received by the editors March 27, 1947.

[1] By a universal sentence we mean here a sentential function without any bound variables ("for all x," "there exists an x." and so on)—or a sentence obtained from such a sentential function by prefixing universal quantifiers. We do not impose the condition that a Boolean algebra must contain at least two elements.