UNSYMMETRICAL APPROXIMATION OF IRRATIONAL NUMBERS

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1. Introduction. In a recent paper, B. Segre[1] showed that for any $\tau \geqq 0$ an irrational number ξ can be approximated by infinitely many fractions A/B in such a way that

$$-\frac{1}{(1+4\tau)^{1/2}B^2} < \frac{A}{B} - \xi < \frac{\tau}{(1+4\tau)^{1/2}B^2}.$$

For $\tau = 0$, this places A/B to the left of ξ and within a distance $1/B^2$ from it. This type of approximation was known to be possible, since alternate convergents to the continued fraction representing ξ satisfy this condition. For $\tau = 1$, the inequality becomes

$$-\frac{1}{5^{1/2}B^2} < \frac{A}{B} - \xi < \frac{1}{5^{1/2}B^2},$$

so that we have the classical theorem of Hurwitz.[2] For other values of τ, approximations from both sides are permitted, but the errors allowed on the two sides are different; hence the term unsymmetrical approximation. The result here was new, and is so related to Hurwitz's inequality that one side is strengthened and the other weakened.

Notice that the result for $\tau > 1$ is weaker than the result for $\tau < 1$. For suppose that $\tau > 1$, and apply the theorem with τ replaced by $1/\tau$ to the irrational number $-\xi$. In this way, the permissible errors on the right and left are interchanged, and we see that ξ has infinitely many approximations A/B satisfying

$$-\frac{1}{(\tau^2+4\tau)^{1/2}B^2} < \frac{A}{B} - \xi < \frac{\tau}{(\tau^2+4\tau)^{1/2}B^2},$$

which is stronger than the original inequality. *It is therefore sufficient to prove Segre's theorem for* $0 \leqq \tau \leqq 1$.

Segre's proof depends on considering whether certain regions con-

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[1] B. Segre, *Lattice points in infinite domains, and asymmetric Diophantine approximations*, Duke Math. J. vol. 12 (1945) pp. 337–365.

[2] A. Hurwitz, *Über die angenäherte Darstellung der Irrationalzahlen durch rationale Brüche*, Math. Ann. vol. 39 (1891) pp. 279–284.