ON AVERAGES OF NEWTONIAN POTENTIALS

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1. **Introduction.** Averages (mean-values) have proved extremely useful in the investigation of properties of potential functions [**2**, **3**, **4**, **7**, **9**, **11**];[1] to a great extent this has been due to the fact that averages of potential functions are themselves *smoother* potential functions.

It is the purpose of this note to exhibit the relations between the mass distribution $\sigma(E)$ associated with the potential function $A[U(x, y)]$, which is an average of a potential function $U(x, y)$, and the mass distribution $\mu(E)$ associated with $U(x, y)$. In a general sense it is proved that $\sigma(E)=A[\mu(E)]$ and that the density $D\sigma(x, y)$ of $\sigma(E)$ is the corresponding average of the density $D\mu(x, y)$. Precise statements of these results are contained in §4 below.

It should be noted that Thompson has investigated the problem noted above [**11**]; except for an error in the statement of his most general result (given without proof), his results are substantially those contained here. However, whereas Thompson's method depends upon a discussion of the interchange of the order of integration in iterated Radon-Stieltjes integrals, the method of this paper depends upon the use of approximations to potentials by means of smoother potentials. Both for the sake of completeness and to point up the difference of the two methods, a proof of Thompson's (corrected) general result (which is Theorem 3 of this note), based upon Thompson's own method, is given in §4 below.

2. **Notation and definitions.** Let F be a closed bounded set in the x, y-plane, and let $\mu(e)$ be an arbitrary *distribution of positive mass* on F, that is, $\mu(e)$ is defined for all Borel sets e in F such that (i) $\mu(e) \geqq 0$, (ii) $\mu(\sum_{i=1}^{\infty} e_i)=\sum_{i=1}^{\infty}\mu(e_i)$, for each sequence $\{e_i\}$ of mutually disjoint Borel sets contained in F [**6**, p. 25]. The distribution $\mu(e)$ is said to be of *finite total amount* if $\mu(e)$ is uniformly bounded for all e in F.

A distribution $\mu(e)$ may be extended so as to be defined for all Borel sets E in the plane by means of the definition [**3**, p. 227] $\mu(E)\equiv\mu(E\cdot F)$. It is apparent that $\mu(E)$ satisfies (i) and (ii) above. In this paper it is assumed that all distributions have been thus extended to all Borel sets in the plane, although, strictly speaking, each distribution had

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[1] Numbers in brackets refer to the bibliography at the end of the paper.