ON THE INTERIOR OF THE CONVEX HULL OF A EUCLIDEAN SET

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In this note we shall prove for each positive integer n the following theorem Δ_n concerning convex sets in an n-dimensional euclidean space.

THEOREM Δ_n. *Any point interior to the convex hull of a set E in an n-dimensional euclidean space is interior to the convex hull of some subset of E containing at most $2n$ points.*

This theorem is similar to the well known result that any point in the convex hull of a set E in an n-dimensional euclidean space lies in the convex hull of some subset of E containing at most $n+1$ points [1, 2].[1] In these theorems the set E is an arbitrary set in the space. The convex hull of E, denoted by $H(E)$, is the set product of all convex sets in the space which contain E.

A euclidean subspace of dimension $n-1$ in an n-dimensional euclidean space will be called a plane. Every plane in an n-dimensional euclidean space separates its complement in the space into two convex open sets, called open half-spaces, whose closures are convex closed sets, called closed half-spaces. If each of the two open half-spaces bounded by a plane L intersects a given set E, then L is said to be a separating plane of E; otherwise L is said to be a nonseparating plane of E.

In order to prove our sequence of theorems we shall make use of the following result: A point i is interior to the convex hull of a set E in an n-dimensional euclidean space if and only if every plane through i is a separating plane of E [1].

We prove our sequence of theorems by induction. The proof of Theorem Δ_1 is trivial and will be omitted. Now suppose that Theorem Δ_{n-1} is true for an integer $n>1$. We shall show that Theorem Δ_n is also true. To this end let i be a point interior to the convex hull of a set E in an n-dimensional euclidean space. We are to demonstrate that i is interior to the convex hull of some subset P of E containing at most $2n$ points.

First we show that i is interior to the convex hull of some finite subset Q of E. Since i is interior to $H(E)$, it is interior to a simplex

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[1] Numbers in brackets refer to the references cited at the end of the paper.