SOME REMARKS ON THE THEORY OF GRAPHS

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The present note consists of some remarks on graphs. A graph G is a set of points some of which are connected by edges. We assume here that no two points are connected by more than one edge. The complementary graph G' of G has the same vertices as G and two points are connected in G' if and only if they are not connected in G.

A special case of a theorem of Ramsey can be stated in graph theoretic language as follows:

There exists a function $f(k, l)$ of positive integers k, l with the following property. Let there be given a graph G of $n \geqq f(k, l)$ vertices. Then either G contains a complete graph of order k, or G' a complete graph of order l. (A complete graph is a graph any two vertices of which are connected. The order of a complete graph is the number of its vertices.)

It would be desirable to have a formula for $f(k, l)$. This at present we can not do. We have however the following estimates:

THEOREM I. *Let* $k \geqq 3$. *Then*

$$2^{k/2} < f(k, k) \leqq C_{2k-2,k-1} < 4^{k-1}.$$

The second inequality of Theorem I was proved by Szekeres,[1] thus we only consider the first one. Let $N \leqq 2^{k/2}$. Clearly the number of different graphs of N vertices equals $2^{N(N-1)/2}$. (We consider the vertices of the graph as distinguishable.) The number of different graphs containing a given complete graph of order k is clearly $2^{N(N-1)/2}/2^{k(k-1)/2}$. Thus the number of graphs of $N \leqq 2^{k/2}$ vertices containing a complete graph of order k is less than

$$(1) \qquad C_{N,k}\frac{2^{N(N-1)/2}}{2^{k(k-1)/2}} < \frac{N^k}{k!}\,\frac{2^{N(N-1)/2}}{2^{k(k-1)/2}} < \frac{2^{N(N-1)/2}}{2}$$

since by a simple calculation for $N \leqq 2^{k/2}$ and $k \geqq 3$

$$2N^k < k!2^{k(k-1)/2}.$$

But it follows immediately from (1) that there exists a graph such that neither it nor its complementary graph contains a complete subgraph of order k, which completes the proof of Theorem I.

The following formulation of Theorem I might be of some interest:

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[1] P. Erdös and G. Szekeres, Compositio Math. vol. 2 (1935) pp. 463–470.