ON THE STRUCTURE OF LINEAR GRAPHS

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Introduction. If the numbers of vertices and edges of a (linear) graph are suitably restricted, it is to be expected that something can be said about the configurations which the graph contains. As far as we know the first result in this direction is due to Turán.[1] He proved that a graph with kn vertices and $C_{k,2}n^2+1$ edges always contains a complete graph of order $k+1$. We shall here prove one such theorem (which arose originally out of a topological problem),[2] and then list (without proofs) several other theorems and conjectures of this nature.

Notations. For the present purposes, a *graph* is simply a finite set of "*vertices*," together with an assignment of certain pairs of vertices (possibly none) as being "*edges*." Two vertices in an edge are said to be *joined*; the *order* of a vertex is the number of vertices to which it is joined. The *complementary* graph G^* to a graph G has the same vertices as G, but two vertices are joined in G^* if and only if they are *not* joined in G. A *complete graph of order k* is a graph having k vertices, every two of which are joined. When $k=3$, this configuration is called simply a triangle. If E is any set, $|E|$ denotes the cardinal number of E. For any real number x, $[x]$ denotes the greatest integer not greater than x, and $[x]^*$ the least integer not less than x. We write $l_1(x)=\ln(x)$, $l_2(x)=\ln(\ln(x))$, and generally $l_r(x)=\ln(l_{r-1}(x))$. Letters like m, n, p, k, N, r, and so on, usually denote positive integers, and ϵ always denotes a positive number less than 1.

THEOREM. *Given ϵ and an integer $r\geqq 2$, there exists $n_0(\epsilon, r)$ such that, for every $n>n_0$, every linear graph having n vertices and fewer than $(1/2(r-1)-\epsilon)n^2$ edges contains r mutually exclusive groups of k vertices each, for some $k\geqq(l_{r-1}(n))^{1/2}$, such that no two vertices in different groups are joined.*

The proof will go by induction over r. First we need a combinatorial lemma.

LEMMA. *Given N subsets $Q_1, Q_2, \cdots, Q_N$ (not necessarily all distinct)*

Received by the editors March 20, 1946.

[1] In fact Turán determined for every k and n the maximum number of edges a graph of n vertices can have without containing a complete graph of k vertices (Matematikai és physikai lapok (1941)) (in Hungarian).

[2] See A. H. Stone, *Connectedness and coherence*, Annals of Mathematics Studies.