ON FRACTIONAL DERIVATIVES OF UNIVALENT FUNCTIONS[1]

H. P. ATKINS

It has been shown by F. Marty [6][2] that if $f(z)$ is analytic and univalent in the unit circle, $f(0)=0$, and $f'(0)=1$, and if the Bieberbach conjecture [2] that $|f^{(n)}(0)| \leqq n\cdot n!$ is assumed when $n>3$, then

$$(1) \qquad |f^{(n)}(z)| \leqq n!(n+r)(1-r)^{-n-2}, \qquad n = 0, 1, 2, 3, \cdots,$$

where $|z|=r$, and that equality is attained for real positive z by the function $f(z)=z(1-z)^{-2}$. For $n=0$ and $n=1$ the inequality reduces to the well known relations obtained by Pick in evaluating the constant in the Verzerrungssatz of Koebe (see [2]).

The purpose of this paper is to generalize this relation to include fractional derivatives and integrals. The bound obtained will be expressed in terms of the ratio of the incomplete to the complete beta function, defined for p and q real and one or the other positive by the equations

$$I_r(p, q) = \frac{1}{B(p, q)} \int_0^r x^{p-1}(1-x)^{q-1}dx, \qquad p > 0,$$

$$I_r(p, q) = 1 - \frac{1}{B(p, q)} \int_r^1 x^{p-1}(1-x)^{q-1}dx, \qquad q > 0,$$

which are equivalent if both p and q are positive. Two separate definitions of the fractional derivative will be found useful; these may be shown equivalent for the values for which both are defined. For $a<0$ the Abel-Riemann definition [1, 9] is more easily applied:

$${}_0D_z^a f(z) = \frac{1}{\Gamma(-a)} \int_0^z f(w)(z-w)^{-a-1}dw, \qquad a < 0,$$

$${}_0D_z^a f(z) = \frac{d^m}{dz^m}\, {}_0D_z^{a-m} f(z), \qquad m - 1 \leqq a < m,$$

where m is a positive integer. For $a \geqq 0$ the Laurent definition [4] is more satisfactory:

Presented to the Society, August 23, 1946; received by the editors April 17, 1946.

[1] The material of this paper forms part of a thesis, prepared under Professor W. Seidel, to be presented to the Graduate School of The University of Rochester for the degree Doctor of Philosophy.

[2] Numbers in brackets refer to the bibliography.