ON UNIFORM SPACES AND TOPOLOGICAL ALGEBRA

G. K. KALISCH

There are several papers which deal with some generalization of metric spaces, for instance, Price [1],[1] Krull's general valuation theory [2], Hyers' pseudonorm [3], and so on. We show in the present paper that spaces possessing a generalized metric (that is, the distance is an element of a partially ordered abelian vector group) are uniform spaces, and that, conversely, any uniform space is a generalized metric space. We also show that topological groups possess right invariant generalized metrics, and, therefore, topological linear spaces, rings, and fields have generalized nǒrms. Although it would be possible to construct a theory based on entirely arbitrary partially ordered abelian semigroups, in view of the construction used in the proof of Theorem 2, it is sufficient to consider only vector groups whose components are real numbers and which are ordered lexicographically. This restriction is not very serious, however, as can be seen from [4] and [5].

The elements of the vector groups we are considering are real numbers indexed according to a partially ordered directed set I. Let i, j, k be elements of I, then:

I (1) If $i \leqq j$, $j \leqq i$ then $i=j$.

(2) If $i \leqq j$, $j \leqq k$ then $i \leqq k$.

(3) Given i and j there exists k such that $k \geqq i$, $k \geqq j$.

We shall denote the elements of G, our vector group, by (r_i), r_i a real number, i an element of I. We define the order relation in G as follows: $g=(r_i) \leqq g'=(r_i')$ in case, if $r_j > r_j'$ for some j, there is some k such that $k<j$, $r_k<r_k'$, and $r_m \leqq r_m'$ for all $m<k$. A space will be called a *generalized metric space* in case there is given a function (s, t) defined on $S \times S$ with $s \in S$, $t \in S$, with values in a partially ordered vector group G as described above, and satisfying the following:

II (1) $(s, t) \geqq 0$; $(s, t)=0$ if and only if $s=t$.

(2) $(s, t)=(t, s)$.

(3) $(s, t) \leqq (s, r)+(r, t)$.

A generalized metric space will be a topological space, and the generalized metric will define a uniform structure in S in case we define neighborhoods or convergence properly. Owing to the fact that G is partially ordered it is possible to topologize G by defining neighbor-

Received by the editors May 16, 1946, and, in revised form, June 4, 1946.

[1] Numbers in brackets refer to the bibliography at the end of the paper.