RING AND LATTICE CHARACTERIZATIONS OF COMPLEX HILBERT SPACE

SHIZUO KAKUTANI[1] AND GEORGE W. MACKEY

Introduction. In an earlier paper [1][2] by the authors it was suggested that at least the ring characterization of real Hilbert space given therein might be extended to the complex case by making use of a device employed by B. H. Arnold [2] in so extending a theorem of Eidelheit. It is the purpose of the present note to show that this can indeed be done not only for the ring characterization but for the lattice one as well.

The difficulty in the complex case is that the complex field admits a great many discontinuous automorphisms. It is overcome by making use of the device of Arnold mentioned above to show that in the infinite-dimensional case only continuous automorphisms present themselves (see Lemma 2 below). It is shown by an example that the infinite-dimensionality is essential and that accordingly the theorems of [1] cannot be extended to the complex case in quite their full generality.

1. Two preliminary lemmas.

Lemmas 1 and 2 below constitute our formulation of Arnold's argument.

LEMMA 1. *Let X be an infinite-dimensional normed linear space (real or complex). Then there exists an infinite sequence $x_1, x_2, x_3, \cdots$ of elements of X such that given any bounded infinite sequence $\lambda_1, \lambda_2, \lambda_3, \cdots$ of scalars there exists a member l of the conjugate space $\overline{X}$ of X such that $l(x_i) = \lambda_i$ for $i = 1, 2, \cdots$.*

PROOF. As is well known, it is possible to construct infinite sequences $x_1, x_2, \cdots; l_1, l_2, \cdots$ where each x_i is in X and each l_i is in $\overline{X}$ so that $l_i(x_j) = \delta_i^j$ for $i, j = 1, 2, \cdots$. Furthermore, it is clear that we may suppose that $\|l_i\| = 1/2^i$. Let $\lambda_1, \lambda_2, \cdots$ be an arbitrary bounded sequence of scalars with $\text{l.u.b.}|\lambda_i| = M$. For each

Received by the editors January 28, 1946, and, in revised form, March 21, 1946.

[1] It should be pointed out that Professor Kakutani is in no way responsible for the appearance of the results of this paper in their present form. They are the result of a continuation of our joint work based upon a suggestion which Professor Kakutani made in a letter just before he left for Japan and before the final draft of our earlier paper had been written. The second named author is entirely responsible for postponing the following up of this suggestion and publishing the results separately in the present paper.

[2] Numbers in brackets refer to the bibliography.