BLOCH'S THEOREM FOR REAL VARIABLES

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The famous theorem of Bloch states that every function

$$F(z) = z + a_2z^2 + \cdots \tag{1}$$

of the complex variable z in the unit-circle $|z| \leqq 1$ maps some subdomain of the unit-circle univalently onto a circle of a fixed radius whose size is independent of $f(z)$. The purpose of the present paper is to point out the following generalization to n real variables, $n \geqq 2$.

THEOREM 1. *Corresponding to any integer n, $n=1, 2, 3, \cdots$, and any positive constant K, $K>0$, there exists a positive radius*

$$R_0 = R_0(n, K), \qquad R_0 > 0,$$

having the following property:

If the real functions

$$u_i = f_i(x_1, \cdots, x_n), \qquad i = 1, 2, \cdots, n, \tag{2}$$

are defined and solutions of the Laplace equation

$$\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0 \tag{3}$$

in a neighborhood of the sphere

$$x_1^2 + \cdots + x_n^2 \leqq 1, \tag{4}$$

if their Jacobian

$$J(x_1, \cdots, x_n) = \frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)} \tag{5}$$

satisfies the decisive relation

$$\sum_{i,j=1}^{n} \left(\frac{\partial f_i}{\partial x_j}\right)^2 \leqq K \cdot |J(x_1, \cdots, x_n)|^{2/n} \tag{6}$$

in (4), *and if*

$$J(0, \cdots, 0) = 1, \tag{7}$$

then there exists in (4) *an open set S such that the functions* (2) *are a one-*

Received by the editors May 6, 1946.