APPROXIMATE ISOMETRIES

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In a recent paper [1][1] Hyers and Ulam formulated the problem of approximate isometries. Thus if E_1 and E_2 are metric spaces, a transformation T on E_1 to E_2 is an ϵ isometry if $|d_1(T(x), T(x')) - d(x, x')| < \epsilon$, for all x, x' in E_1. These authors analyzed the ϵ isometries defined on a complete abstract Euclidean space E and showed that if T maps E onto itself and $T(\theta)=\theta$, then there exists an isometry [2, p. 165], U, of E onto E such that $\|T(x)-U(x)\|<10\epsilon$. The analysis depends on the properties of the scalar product. In the present work we show, first, that similar results hold when $E_1=E_2=L_r(0, 1)$, $1<r<\infty$, though, except of course for $r=2$, a scalar product no longer exists. It is shown further that it is sufficient that E_2 belong to a restricted class of uniformly convex Banach spaces and that E_1 be a Banach space.

THEOREM 1. *Let $T(x)$ be an ϵ isometry of $L_r(0, 1)$, $1<r<\infty$, into itself with $T(\theta)=\theta$. Then $U(x)=L_{n\to\infty}T(2^n x)/2^n$ exists for each x and $U(x)$ is an isometric, linear transformation.*

Our fundamental assumption is that

$$| \| T(x) - T(x') \| - \| x - x' \| | < \epsilon, \qquad T(\theta) = \theta. \tag{1.01}$$

The following inequality is due to Clarkson [3, 4],

$$\|\alpha + \beta\|^p + \|\alpha - \beta\|^p \leqq 2(\|\alpha\|^q + \|\beta\|^q)^{p-1}, \tag{1.02}$$

where here and later we understand that

$$p = \sup (r/(r-1)) \geqq 2 \geqq q = \inf (r, r/(r-1)).$$

Let

$$2\alpha = T(x),\ 2\beta = T(x) - T(2x).$$

Then

$$\begin{aligned} &\|T(x)-T(2x)/2\|^p \\ &\leqq 2^{1-q(p-1)}(\|T(x)\|^q+\|T(x)-T(2x)\|^q)^{p-1}-\|T(2x)/2\|^p \\ &\leqq (\|x\|+\epsilon)^p-(\|x\|-\epsilon/2)^p. \end{aligned} \tag{1.03}$$

If $\|x\| \leqq \epsilon$ then the right-hand side of equation (1.03) is surely in-

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[1] Numbers in brackets refer to the Bibliography.