DERIVATIVES OF COMPOSITE FUNCTIONS

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1. Introduction. The object of this note is to show the relation of the Y polynomials of E. T. Bell [1],[1] first to the formula of diBruno for the nth derivative of a function of a function, then to the more general case of a function of many functions. The subject belongs to the algebra of analysis in the sense of Menger [4]; all that is asked is the relation of the derivative of the composite function to the derivatives of its component functions when they exist and no questions of analysis are examined.

2. Function of a single function. Following Dresden [3], take the composite function in the form:

$$F(x) = f[g(x)]; \tag{1}$$

and for convenience write:

$$D_x^s F(x) \equiv F_s, \qquad [D_u^s f(u)]_{u=g(x)} \equiv f_s, \qquad D_x^s g(x) \equiv g_s,$$

with $D_x = d/dx$.

Then, the first few derivatives of $F(x)$ are as follows:

$$F_1 = f_1 g_1, \qquad F_2 = f_1 g_2 + f_2 g_1^2, \qquad F_3 = f_1 g_3 + 3 f_2 g_2 g_1 + f_3 g_1^3.$$

If these are generalized to

$$F_n = \sum_{i=1}^{n} F_{n,i} f_i \tag{2}$$

the coefficients $F_{n,i}$ are dependent only on the derivatives g_1 to g_i, and hence may be determined by specialization of f. A convenient choice used by Schlömilch [6] is $f(g) = \exp (ag)$, so that

$$f_i = a^i \exp ag$$

and

$$e^{-ag} F_n = e^{-ag} D_x^n e^{ag} = \sum_{i=1}^{n} F_{n,i}(g_1, \cdots, g_i) a^i, \tag{3}$$

a generating identity for the $F_{n,i}$, closely related to the definition equation of the Y polynomials (Bell, loc. cit. p. 269), namely

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.