NOTE ON THE KUROSCH-ORE THEOREM

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1. Introduction. The Kurosch-Ore theorem[1] asserts that if an element of a modular lattice has two decompositions into irreducibles, then each irreducible of one decomposition may be replaced by a suitably chosen irreducible from the other decomposition. It follows that the number of irreducibles in the two decompositions is the same.

The purpose of the present note is to study the manner in which the irreducibles of two decompositions can replace one another. Now from the Kurosch-Ore theorem it is not even clear that each irreducible of one decomposition is suitable for replacing some irreducible of the other decomposition. However, this follows from the following precise theorem:

THEOREM 1. *Let a be an element of a modular lattice and let $a=q_1\cap \cdots \cap q_n=q_1'\cap \cdots \cap q_n'$ be two reduced decompositions into irreducibles. Then the q's may be renumbered in such a way that*

$$a = q_1\cap \cdots \cap q_{i-1}\cap q_i'\cap q_{i+1}\cap \cdots \cap q_n, \qquad i = 1, \cdots, n.$$

Along the same line of ideas, the following theorem on simultaneous replacement is also proved.

THEOREM 2. *Let a be an element of a modular lattice and let $a=q_1\cap \cdots \cap q_n=q_1'\cap \cdots \cap q_n'$ be two reduced decompositions into irreducibles. Then for each q_i, there exists q_j' such that q_j' can replace q_i in the first decomposition and q_i can replace q_j' in the second decomposition.*

On the other hand, an example is given which shows that, in general, it is impossible to renumber the q's in such a way that simultaneously q_i may replace q_i' and q_i' replace q_i.

As the principal tool in the investigation we introduce the concept of a *superdivisor* r of an element a. r has the fundamental property that its crosscut with any proper divisor of a is never equal to a. The superdivisors of a are closed under crosscut and indeed form a dual-ideal $\mathfrak{r}_a$ which properly divides a.

A surprising by-product of the investigation is the fact that in a

Presented to the Society, September 8, 1942, under the title *On the decomposition theory of modular lattices*; received by the editors April 9, 1946.

[1] A simple proof is given in Birkhoff [1, p. 54]. Numbers in brackets refer to the references cited at the end of the paper.