THE KLINE SPHERE CHARACTERIZATION PROBLEM

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The object of this paper is to give a solution to the following problem proposed by J. R. Kline: Is a nondegenerate, locally connected, compact continuum which is separated by each of its simple closed curves but by no pair of its points homeomorphic with the surface of a sphere? The answer is in the affirmative.

A solution to the Kline problem gives a characterization of a simple closed surface. Partial solutions of this problem have been made by Hall [1, 2][1] and Jones [3]. Other characterizations of a simple closed surface have been given by Kuratowski [4], Zippin [5, 6], Wilder [7] and Claytor [8]. Previous to the giving of these characterizations, Moore gave [9] two sets of axioms, each set of which characterized a set topologically equivalent to a plane.

DEFINITION. We say that M disrupts X from Y in D if there is an arc from X to Y in D but each such arc contains a point of M.

We shall make use of the following lemma.

LEMMA. *Suppose that space is locally connected and cannot be separated by the omission of any pair of its points, that the boundary of the connected domain D is equal to the sum of the mutually exclusive sets M, N and E, each of which contains a point which is accessible from D, and that D' is a connected subdomain of D such that no point of D either disrupts D' from $E+M$ in $D+E+M$ or disrupts D' from $E+N$ in $D+E+N$. Then there is an open arc from M to N in D that does not disrupt D' from E in $D+E$.*

PROOF. Consider the arc AB in $D+B$ from a point A of D' to a point B of E. Let W_1 be the set of all points P of AB such that there is an open arc from P to E in D that does not intersect some open arc from M to N in D. Assume that the first point R of AB in the order from A to B on the closure of W_1 does not belong to D'.

If R disrupts D' from E in $D+E$, there are an arc from D' to M in $D+M-R$ and an arc from D' to N in $D+N-R$. In the sum of these two arcs plus D' there is an open arc from M to N in D which does not intersect RB. This is contrary to the definition of R. Hence, R does not disrupt D' from E in $D+E$.

Let $A'B'$ be an arc in $D+B'-R$ from a point A' of D' to a point

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[1] Numbers in brackets refer to the references cited at the end of the paper.