LATTICES, EQUIVALENCE RELATIONS, AND SUBGROUPS

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1. **Introduction.** A lattice[1] is a set of elements $A, B, \cdots$ together with a relation $\leqq$ defined between some or all pairs of the elements such that $A \leqq A$ for each A; if $A \leqq B$ and $B \leqq A$ then $A = B$; if $A \leqq B$ and $B \leqq C$ then $A \leqq C$; and each pair of elements A and B have a least upper bound $A \cup B$ and a greatest lower bound $A \cap B$.

An equivalence relation[2] R' is a relation $\sim$ defined between some or all pairs of elements ("points") of a set $\mathfrak{S}$ such that $p \sim p$ for every point p; if $p \sim q$ then $q \sim p$; if $p \sim q$ and $q \sim r$ then $p \sim r$. If more than one relation is under discussion, we may specify the one used at the moment by writing "$p \sim q$ in R'."

It is well known that the collection of all equivalence relations on a given set $\mathfrak{S}$ forms a lattice. Ore [**4**] has characterized lattices which are isomorphic to the lattice $\mathfrak{L}^*$ of all equivalence relations on some set $\mathfrak{S}$. One could go farther and ask what lattices are isomorphic to sublattices of $\mathfrak{L}^*$. Our answer to this is: *any* lattice is isomorphic to a sublattice of the lattice of all equivalence relations on some set; more concisely, any lattice is a lattice of equivalence relations.

Garrett Birkhoff has shown [**1**] that any lattice of equivalence relations is isomorphic to a lattice of subgroups. Therefore the result stated in the previous paragraph implies: any lattice is isomorphic to a sublattice of the lattice of all subgroups of a suitable group.

2. **Outline.** The first and larger part of the paper relates lattices and equivalence relations. Since the formal construction and proof in this part are somewhat lengthy and complicated, we first outline the main steps and indicate the motivation.

A lattice $\mathfrak{L}$ is given; we wish to show that there is some set $\mathfrak{S}$, and some sublattice $\mathfrak{L}'$ of the lattice $\mathfrak{L}^*$ of all equivalence relations on $\mathfrak{S}$, such that $\mathfrak{L}$ and $\mathfrak{L}'$ are isomorphic.

We shall take $\mathfrak{S}$ as the union of disjoint subsets $\mathfrak{A}, \mathfrak{B}, \cdots$, one for each element $A, B, \cdots$ of $\mathfrak{L}$. The equivalence relations $A', B', \cdots$ corresponding to $A, B, \cdots$ must be chosen in such a manner that

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[1] Sometimes called a *structure*. For a thorough discussion of lattices, see Birkhoff [**2**]. Numbers in brackets refer to the references cited at the end of the paper.

[2] For a discussion of the relevant properties of equivalence relations (or *congruence relations* as they are sometimes called), see Birkhoff [**1**], Dubreil and Dubreil-Jacotin [**3**], or Ore [**4**].