NOTE ON A THEOREM OF MURRAY

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1. Introduction. In a recent paper[1] [1][2] Murray has shown that in any reflexive separable Banach space $\mathfrak{B}$ every closed subspace $\mathfrak{M}$ admits what he calls a quasi-complement, that is, a second closed subspace $\mathfrak{N}$ such that $\mathfrak{M} \cap \mathfrak{N} = 0$ and such that $\mathfrak{M} \dotplus \mathfrak{N}$, the smallest subspace containing both $\mathfrak{M}$ and $\mathfrak{N}$, is dense in $\mathfrak{B}$. It is the purpose of this note to give a simpler proof of the following somewhat more general theorem.

THEOREM. *Let $\mathfrak{B}$ be a separable normed linear space (not necessarily reflexive or even complete) and let $\mathfrak{M}$ be a closed subspace of $\mathfrak{B}$. Then there exists a second closed subspace $\mathfrak{N}$ such that $\mathfrak{M} \cap \mathfrak{N} = 0$ and $\mathfrak{M} \dotplus \mathfrak{N}$ is dense in $\mathfrak{B}$.*

In proving this theorem it is convenient to make use of the notion of closed subspace of a linear system discussed at length in Chapter III of [2]. We repeat the necessary definitions here. A linear system X_L is an abstract linear space X together with a linear subspace L of the space X^* of all linear[3] functionals defined on X. If $l(x) = 0$ for all l in L implies that $x = 0$ (that is, if L is total) we say that X_L is a regular linear system. If M is a subspace of X $[L]$ we denote by M' the set of all l in L $[x$ in $X]$ such that $l(x) = 0$ for all x in X $[l$ in $L]$. It is clear that $M \subseteq N$ implies $N' \subseteq M'$ and that $M'' \supseteq M$. Since $M''' = (M'')' \subseteq M'$ and since $M''' = (M')'' \supseteq M'$ it follows that $M' = M'''$ and hence that $M = M''$ if and only if M is of the form N'. A subspace having either and hence both of these properties is said to be closed. We observe that the operation $'$ sets up a one-to-one inclusion inverting correspondence between the closed subspaces of X and L respectively.

2. Two lemmas. The proof of the theorem is based essentially on the following lemma.

LEMMA 1. *Let X_L be a regular linear system such that both X and L are $\aleph_0$ dimensional, that is, have $\aleph_0$ independent generators. Then if M*

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[2] Numbers in brackets refer to the bibliography.

[3] By linear we mean additive and homogeneous.