ON THE THEOREM OF FEJÉR-RIESZ

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1. Statement of results. Let

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots \tag{1}$$

be a function regular for $|z| \leqq 1$. The well known inequality of Fejér and Riesz asserts that

$$\int_D |f(z)||dz| \leqq \frac{1}{2}\int_C |f(z)||dz|, \tag{2}$$

where C is the circumference $|z| = 1$, and D any of its diameters.[1]

For $f(z) = F'(z)$, the inequality (2) takes the form

$$\int_D |F'(z)||dz| \leqq \frac{1}{2}\int_C |F'(z)||dz|, \tag{3}$$

which shows that the total variation of $F(z)$ along D does not exceed half of the total variation of F along C. In this form the inequality remains valid for harmonic functions. Let $z = \rho e^{i\theta}$. If $U(z) = U(\rho, \theta)$ is harmonic for $|z| \leqq 1$, the total variation of F along D does not exceed half of the total variation of F along C.[2] In symbols,

$$\int_D |U_\rho| d\rho \leqq \frac{1}{2}\int_C |U_\theta| d\theta. \tag{4}$$

Let $V(z) = V(\rho, \theta)$ be the harmonic function conjugate to U. In (4) we may replace U_ρ by $\rho^{-1}V_\theta$. Writing $U_\theta = u$, $V_\theta = v$, we obtain an equivalent form of the inequality (4), namely

$$\int_D \left|\frac{v(z)}{z}\right||dz| \leqq \frac{1}{2}\int_C |u(z)||dz|. \tag{5}$$

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[1] L. Fejér and F. Riesz, *Ueber eine funktionentheoretische Ungleichung*, Math. Zeit. vol. 11 (1921) pp. 305–314.

[2] The inequality (4), with 1/2 on the right replaced by an undetermined constant A, was first proved by B. N. Prasad, *On the summability of power series and the bounded variation of power series*, Proc. London Math. Soc. vol. 35 (1933) pp. 407–424. That $C = 1/2$ was shown in F. Riesz, *Eine Ungleichung für harmonische Funktionen*, Monatshefte für Mathematik und Physik vol. 43 (1936) pp. 401–406, and A. Zygmund, *Some points in the theory of trigonometric and power series*, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 586–617, especially p. 599.