

40. A. R. Schweitzer: *On the genesis of number systems. II.*

In continuation of the preceding paper, other developments of number systems are: (2) in terms of reflexive, symmetric and transitive relations $\alpha\beta R\gamma\delta(\alpha+\beta=\gamma+\delta)$ and $\kappa\lambda P\mu\nu(\kappa\times\lambda=\mu\times\nu)$ in analogy with the author's relation $\alpha\beta K\gamma\delta$ (ibid. p. 394) and (3) as an elaboration of linear order based on the author's system 1R_1 (ibid. p. 378). In system (2) the author employs the binary relational symbol $\alpha\beta RP\gamma\delta$ to express $R\alpha\beta=P\gamma\delta$ or $\alpha+\beta=\gamma\times\delta$; $\alpha\beta RR\gamma\delta$ and $\kappa\lambda PP\mu\nu$ are assumed respectively equivalent to $\alpha\beta R\gamma\delta$ and $\kappa\lambda P\mu\nu$. Further assumptions are: 1. $\alpha\beta RP\gamma\delta$ implies $\gamma\delta PR\alpha\beta$. 2. $\alpha\beta XY\lambda\mu$, $\lambda\mu YZ\gamma\delta$ imply $\alpha\beta XZ\gamma\delta$, where X, Y, Z are on the set (R, P) . 3. There exist uniquely in S the distinct elements ω (zero) and ϵ (unity) such that $\alpha\omega RP\alpha\epsilon$ for any α in S . 4. $\alpha\omega RP\beta\epsilon$ implies $\alpha=\beta$. 5. For any α, β in S there exist uniquely γ, δ such that $\alpha\beta R\gamma\omega$ and $\alpha\beta P\delta\epsilon$. 6. $\alpha\beta R\xi\omega$ and $\beta\gamma R\eta\omega$ imply $\alpha\eta R\xi\gamma$; and similarly for the relation P . Correspondingly, distributive, inversive and commutative properties are stated. The extension of the preceding system S in analogy with the author's systems nR_n and nK_n ($n=1, 2, 3, \dots$) is discussed. (Received October 19, 1945.)

STATISTICS AND PROBABILITY

41. Will Feller: *Note on the law of large numbers and "fair" games.*

An example is exhibited to show that a game can be "fair" in the sense that the expectation of loss vanishes, and nevertheless such that the probability tends to one that after n trials there will be a positive loss L_n ; the ratio of L_n to the accumulated entrance fees tends to zero as slowly as one pleases. On the other hand, in the classical Petersburg game entrance fees can be determined so that the game becomes fair in the sense that the probable loss or gain will be of smaller order of magnitude than the accumulated entrance fees. (To appear in the Annals of Mathematical Statistics.) (Received October 4, 1945.)

42. Will Feller: *On the normal approximation to the binomial.*

The goodness of the normal approximation to $T_{\lambda,\nu}=\sum_{k=\lambda}^{\nu} C_{n,k}p^k(1-p)^{n-k}$ is studied with particular reference to the practically important cases of small $np(1-p)$ and of comparatively large λ and ν . Limits of the integral are determined which depend quadratically on λ and ν and are such that the integral will approximate $T_{\lambda,\nu}$ from above or from below. The relative error is also studied. In a sense this paper continues a well known series of studies by Serge Bernstein (the latest in *Izvestia Akademiiia Nauk SSR*, 1943). By the departure from the classical, but arbitrary, use of moments unexpected simplifications are obtained which render S. Bernstein's results more accurate and valid under less stringent conditions. (To appear in the Annals of Mathematical Statistics.) (Received October 4, 1945.)

43. P. R. Halmos: *The theory of unbiased estimation.*

Let $F(P)$ be a real-valued function defined on a subset E of the set D of all probability distributions on the real line. A function f of n real variables is an unbiased estimate of F if for every system X_1, \dots, X_n of independent random variables with the common distribution P the expectation of $f(X_1, \dots, X_n)$ exists and equals $F(P)$, for all P in E . Under the assumption that E contains all purely discontinuous distributions, the class of all functions $F(P)$ which possess an unbiased estimate is characterized and all unbiased estimates of each such F are described. It is shown that there is in every case a unique symmetric unbiased estimate which coincides also