ON RIEMANNIAN MANIFOLDS OF FOUR DIMENSIONS[1]

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Introduction. It is well known that in three-dimensional elliptic or spherical geometry the so-called Clifford's parallelism or parataxy has many interesting properties. A group-theoretic reason for the most important of these properties is the fact that the universal covering group of the proper orthogonal group in four variables is the direct product of the universal covering groups of two proper orthogonal groups in three variables. This last-mentioned property has no analogue for orthogonal groups in n (>4) variables. On the other hand, a knowledge of three-dimensional elliptic or spherical geometry is useful for the study of orientable Riemannian manifolds of four dimensions, because their tangent spaces possess a geometry of this kind. It is the purpose of this note to give a study of a compact orientable Riemannian manifold of four dimensions at each point of which is attached a three-dimensional spherical space. This necessitates a more careful study of spherical geometry than hitherto given in the literature, except, so far as the writer is aware, in a paper by E. Study [2].[2] Our main result consists of two formulas, which express two topological invariants of a compact orientable differentiable manifold of four dimensions as integrals over the manifold of differential invariants constructed from a Riemannian metric previously given on the manifold. These two topological invariants have a linear combination which is the Euler-Poincaré characteristic.

1. **Three-dimensional spherical geometry.** We consider an oriented Euclidean space of four dimensions E^4 with the coordinates x_0, x_1, x_2, x_3. In E^4 let S^3 be the oriented unit hypersphere defined by the equation

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1. \tag{1}$$

Three-dimensional spherical geometry is concerned with properties on S^3 which remain invariant under the rotation group (that is, the proper orthogonal group) of E^4 leaving the origin fixed.

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[1] The content of this paper was originally intended to be an illustration in the author's article, *Some new viewpoints in differential geometry in the large*, which is due to appear in this Bulletin. Later it appeared more advisable to publish these results separately, but a comparison with the above-mentioned article, in particular §7, is recommended.

[2] Numbers in brackets refer to the references cited at the end of the paper.