A NOTE ON THE ZEROS OF THE SECTIONS OF A PARTIAL FRACTION

MORRIS MARDEN

1. **Introduction.** If $f(z)$ is a rational function with a total of three distinct zeros and poles, the zeros of its logarithmic derivative may be located as points in the complex plane by aid of the following theorem.

THEOREM 1. *The zeros of the partial fraction*

$$F(z) = \frac{m_1}{z - z_1} + \frac{m_2}{z - z_2} + \frac{m_3}{z - z_3}, \qquad m_1 m_2 m_3 \neq 0,$$

where z_1, z_2 and z_3 are three distinct, noncollinear points, lie at the foci of the conic which touches the line segments (z_2, z_3), (z_3, z_1) and (z_1, z_2) in the points ζ_1, ζ_2 and ζ_3 that divide these segments in the ratio $m_2:m_3$, $m_3:m_1$, and $m_1:m_2$ respectively. If $n = m_1+m_2+m_3 \neq 0$, this conic is an ellipse or hyperbola according as $nm_1m_2m_3 > 0$ or < 0. If $n = 0$, the conic is a parabola whose axis is parallel to the line joining the origin to the point $\nu = m_1z_1+m_2z_2+m_3z_3$.

In the special case $m_1 = m_2 = m_3 = 1$, this theorem was proved geometrically by Bôcher and Grace.[1] In the general case it was first deduced by Linfield as a corollary to the following theorem which in turn was established by the use of line coordinates and polar forms.[2]

THEOREM 2. *The zeros of the partial fraction $F(z) = \sum_{j=1}^{p} m_j/(z - z_j)$ lie at the foci of the curve $C(z_1, z_2, \cdots, z_p; m_1, m_2, \cdots, m_p)$ of class $p-1$ which touches each of the $p(p-1)/2$ line-segments (z_j, z_k) in a point dividing it in the ratio $m_j:m_k$.*

In view, however, of the elementary character of Theorem 1, it

Presented to the Society, April 28, 1945; received by the editors April 23, 1945.

[1] M. Bôcher, Ann. of Math. vol. 7 (1892) pp. 70–76; J. H. Grace, Proc. Cambridge Philos. Soc. vol. 11 (1901) pp. 352–357.

[2] For the case that all $m_j > 0$, see Siebeck, J. Reine Angew. Math. vol. 64 (1864) p. 175; M. Van den Berg, Niew Archief voor Wiskunde vol. 9 (1882) pp. 1–14, 60, vol. 11 (1884) pp. 153–186, vol. 15 (1899) pp. 100–164; J. Juhel-Renjoy, C. R. Acad. Sci. Paris vol. 142 (1906); P. J. Heawood, Quart. J. Math. vol. 38 (1907) pp. 84–107; and M. Fujiwara, Tôhoku Math. J. vol. 9 (1916) pp. 102–108. It is to be observed that, although priority for the theorem when all $m_j > 0$ is usually accorded to Van den Berg, it should rightfully be given to Siebeck. For arbitrary integral m_j, see B. Z. Linfield, Bull. Amer. Math. Soc. vol. 27 (1920) pp. 17–21 and Trans. Amer. Math. Soc. vol. 25 (1923) pp. 239–258.