ON A CHARACTERISTIC PROPERTY OF LINEAR FUNCTIONS

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1. **Introduction.** If the real function $y=g(x)$, defined and continuous in the closed and bounded interval $a \leqq x \leqq b$, or in the open interval $a<x<b$, is linear there,

$$y = px + q,$$

then for all x_0, h, with $h>0$, such that x_0-h and x_0+h lie in the interval of definition, we have

$$g(x_0) = [g(x_0 - h) + g(x_0 + h)]/2. \tag{1}$$

Conversely, if the real function $y=g(x)$, defined and continuous in $a \leqq x \leqq b$ or in $a<x<b$, satisfies (1) for all x_0, h, with $h>0$, such that x_0-h and x_0+h lie in the interval of definition, then [**4**, p. 189][1] $y=g(x)$ is a linear function of x.

If, however, in the converse it is given only that for each x_0, $a<x_0<b$, there exists a positive $h_0=h_0(x_0)$, such that x_0-h_0 and x_0+h_0 lie in the interval of definition, and for which we have

$$g(x_0) = [g(x_0 - h_0) + g(x_0 + h_0)]/2, \tag{2}$$

then the implications are different in the case that $g(x)$ is defined and continuous in the closed and bounded interval and in the case that $g(x)$ is defined and continuous only in the open interval; for in the former case it still follows [**3**, p. 253] that $g(x)$ must be linear, while in the latter case $g(x)$ is not necessarily linear [**3**, pp. 253–255].

A proof of the above result, that if $g(x)$ is defined and continuous in the closed and bounded interval and satisfies (2) then $g(x)$ necessarily is linear, can be given, as we shall show, which applies equally well to characterize, in terms of equalities analogous to (2), classes of functions [**1**] differing, and even topologically distinct [**2**], from the class of linear functions.

2. **Theorem.** We shall establish the following result.

THEOREM. *Let $\{f(x)\}$ be a class of functions defined and continuous in the closed and bounded interval $a \leqq x \leqq b$, and such that for all real (x_1, y_1), (x_2, y_2) with $a \leqq x_1 < x_2 \leqq b$ there is a unique member*

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[1] Numbers in brackets refer to the references cited at the end of the paper.