NOTE ON THE EQUIVALENCE OF NONSINGULAR PENCILS OF HERMITIAN MATRICES

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In a previous paper[1] on the equivalence of nonsingular pencils of Hermitian matrices the restriction was made that the characteristic of the field in which the elements of the matrices lay be zero. It is the purpose of this note to show how this restriction can be removed and that all the results of this paper are true for fields of any characteristic different from two. Since many of the proofs involve division by the integer two, the case of characteristic two is not considered. To save unnecessary repetitions the notations used here conform with those of the previous paper.

If A, B and C, D are two pairs of Hermitian matrices, the pencil $A-\lambda B$ is said to be conjunctively equivalent to the pencil $C-\lambda D$ when there exists a nonsingular matrix P such that $P^*(A-\lambda B)P = C-\lambda D$, where P^* is the conjugate transposed of P. If B is nonsingular, the invariant factors of the pencil $A-\lambda B$ are the same as those of the pencil $AB^{-1}-\lambda E$. Let M be any matrix similar to AB^{-1} so that there exists a nonsingular matrix P such that $P^{-1}AB^{-1}P=M$. Then

$$P^*B^{-1}(A - \lambda B)B^{-1}P = P^*B^{-1}P(M - \lambda E) = RM - \lambda R.$$

Therefore the pencil $A-\lambda B$ is conjunctively equivalent to the pencil $RM-\lambda R$ and, since RM is Hermitian,

$$RM = M^*R. \tag{1}$$

Further, if Q is any nonsingular matrix satisfying

$$QM = M^*Q, \tag{2}$$

it follows from (1) that

$$R = QS, \tag{3}$$

where S is commutative with M. Moreover, if the pencil $RM-\lambda R$ is conjunctively equivalent to the pencil $GM-\lambda G$, so that there exists a nonsingular matrix W such that $W^*(RM-\lambda R)W=G(M-\lambda E)$, the matrix W is commutative with M.

In paper I the matrix M is taken in the Wedderburn canonical

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[1] John Williamson, *The equivalence of non-singular pencils of Hermitian matrices in an arbitrary field*, Amer. J. Math. vol. 57 (1935) pp. 475–490. This paper will be referred to as paper I.