TRANSFINITE RATIONALS

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The standard method of extending a system with an operation to a system with the operation and its inverse depends heavily on the law of cancellation: $ab=ac$ implies $b=c$ (possibly with some limitation on a). The most familiar example is undoubtedly the extension of the integers to the rational numbers. If the law of cancellation fails, not only does the method fail, but the extension cannot be a group. However, one may ask whether some method of extension is available in certain cases where the cancellation law fails but is replaced by something else, for example an order relation. It is the purpose of this paper to present an extension of the system of cardinal numbers (the positive cardinals, excluding zero). This will be an extension of both the cardinal numbers and the positive rational numbers with respect to the operations of addition and multiplication and the relation of ordering. Furthermore, this extension is the smallest extension subject to certain conditions. We shall assume the axiom of choice in the form of the simple ordering of the cardinals.

The means for obtaining the extension is suggested by a treatment of ratios by Eudoxus. (See for instance E. T. Bell, *Development of mathematics*, New York, 1940, p. 61.)

We define first an equivalence relation between ordered pairs of cardinal numbers: $(a, b)\rho(c, d)$ *if and only if* (1) $a>b$ *and* $c>d$, *or* (2) $a=b$ *and* $c=d$, *or* (3) $a<b$ *and* $c<d$.

In terms of the relation ρ we define the fundamental equivalence relation between ordered pairs of cardinal numbers:

DEFINITION. $(a, b)\sim(c, d)$ *if and only if for every pair of cardinal numbers*, m *and* n, $(ma, nb)\rho(mc, nd)$.

The relation $\sim$ is readily shown to be an equivalence relation. Before discussing the equivalence classes defined by this relation we shall state a theorem, assuming for the first two parts that $(a_1, b_1)\sim(a_2, b_2)$.

THEOREM. I. *If* a_1 *and* b_1 *are finite so are* a_2 *and* b_2. II. *If* $a_1<b_1$ *and* b_1 *is infinite, then* $b_1=b_2$. *If* $a_1>b_1$ *and* a_1 *is infinite, then* $a_1=a_2$. III. *If* $a_1<b$, $a_2<b$, *and* b *is infinite, then* $(a_1, b)\sim(a_2, b)$. *If* $a>b_1$, $a>b_2$, *and* a *is infinite, then* $(a, b_1)\sim(a, b_2)$. IV. *If* $(a, a)\sim(b, b)$ *and* a *and* b *are infinite, then* $a=b$.

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