NOTE ON FACTORIZATION IN A QUADRATIC FIELD

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1. **Introduction.** In this note we shall prove certain theorems relating to the "existence" and "uniqueness" of factorization in a quadratic field (cf. §§2 and 3); and shall maintain that the introduction of ideals should be regarded as restoring *existence* rather than *uniqueness* of factorization into primes.

To illustrate this, let us consider first the case of quaternions. Let $x = x_0 + i_1x_1 + i_2x_2 + i_3x_3$ be a *primitive* quaternion, that is, let the coordinates $x_0, \cdots, x_3$ be relative-prime rational integers. Let the norm $Nx = \sum x_i^2$ be factored into a product of rational primes $p_1 \cdots p_s$. Then, by a theorem of Lipschitz,[1] there exist prime quaternions $t', t'', \cdots, t^{(s)}$, of respective norms $p_1, \cdots, p_s$ such that $x = t't'' \cdots t^{(s)}$. This factorization is unique, for any given ordering of the primes $p_1, \cdots, p_s$, except that we can insert unit factors in the trivial way illustrated by the example $t't''t''' = (t'i_1)(i_1t''i_3)(i_3t''') = (-t')(t''i_2)(i_2t''') = \cdots$.

It is proved elsewhere that a similar uniqueness of factorization holds in every system of "generalized quaternions," but that the existence of such a factorization will fail if certain rational primes p_i are not norms.

As is well known there exists a very satisfactory arithmetic of ordinary quaternions, without the necessity of introducing ideals. Nevertheless, factorization of imprimitive quaternions is not unique. For example,

$$6 = (1 - i_1 - i_2)(1 - i_1)(1 + i_1)(1 + i_1 + i_2)$$
$$= (1 - i_1 - i_3)(1 - i_1)(1 + i_1)(1 + i_1 + i_3),$$

where the primes $1 - i_1 - i_2$ and $1 - i_1 - i_3$ do not differ only by unit factors.

Similarly, in the quadratic field $R(\rho)$, where $\rho^2 = -5$, we have

$$6 = (1 + \rho)(1 - \rho) = 2 \cdot 3,$$

where the factors are essentially different prime integers of the field, and hence factorization is not unique. Yet a uniqueness theorem analogous to that for ordinary quaternions holds for the factoriza-

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[1] Lipschitz, Journal de Mathématiques (4) vol. 2 (1886) pp. 373–439; Hurwitz, *Vorlesungen über die Zahlentheorie der Quaternionen*, 1919.