# THE COEFFICIENTS OF UNIVALENT FUNCTIONS

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**1. Introduction.** Let the function

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots, \qquad c_n \text{ real}, \tag{1.1}$$

be regular and convex in the direction of the imaginary axis for $|z| < 1$. Thus each circle $|z| = r$, $0 < r < 1$, is mapped by $f(z)$ into a contour $C_r$ which has the property that straight lines parallel to the imaginary axis cut $C_r$ in at most two points. Since the coefficients are all real, $C_r$ is symmetric about the real axis. For

$$f(re^{i\theta}) = U(r, \theta) + iV(r, \theta)$$

we have $\partial U(r, \theta)/\partial\theta \leqq 0$ for $0 < \theta < \pi$. In other words, $zf'(z)$ is typically real for $|z| < 1$. It is well known [1, 2][1] that the coefficients $c_n$ are bounded, $|c_n| \leqq |c_1|$, $n = 1, 2, \cdots$, and [3] have the representation

$$c_n = \frac{c_1}{n\pi} \int_0^\pi \frac{\sin n\theta}{\sin\theta} d\alpha(\theta) \tag{1.2}$$

where $\alpha(\theta)$ is a nondecreasing function of $\theta$ in $(0, \pi)$ normalized so that

$$\frac{1}{\pi} \int_0^\pi d\alpha(\theta) = 1.$$

A sufficient condition that $f(z)$, given by the series (1.1), be regular and convex in the direction of the imaginary axis for $|z| < 1$ is that the sequence $\{c_n\}$ be monotonic of order 4, a theorem due to L. Fejér [4]. A sequence $\{c_n\}$ is said to be monotonic of order $p$ if each of the differences

$$\Delta^{(k)} c_n = c_n - C_{k,1} c_{n+1} + C_{k,2} c_{n+2} - \cdots + (-1)^k C_{k,k} c_{n+k} \tag{1.3}$$

are non-negative for $k = 0, 1, 2, \cdots, p$; $n = 0, 1, 2, \cdots$. This sufficiency test implies, among other inequalities, that $0 \leqq c_n - c_{n+1}$. This suggests the problem of finding an upper bound for $c_n - c_{n+1}$ for functions $f(z)$ given by (1.1) which are convex in the direction of the imaginary axis for $|z| < 1$. The example $c_1 z(1+z)^{-1}$ shows that the upper bound $2|c_1|$ is sharp. However, if we consider the differences $c_{n-1} - c_{n+1}$ we obtain an inequality which is not so immediately obvious. This inequality is stated in the following theorem.

Presented to the Society, September 17, 1945; received by the editors May 21, 1945.

[1] Numbers in brackets refer to the references cited at the end of the paper.