A NEW CHARACTERISTIC PROPERTY OF MINIMAL SURFACES

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1. Isothermal families on a surface Σ defined by the Monge equation $z=f(x, y)$. We shall present a new characteristic property of minimal surfaces (Theorem 3). We also discover an extensive class of surfaces (Theorem 1), including minimal surfaces and surfaces of revolution.[1] It is shown that the minimal surfaces are the only ones such that every set of parallel planes intersects the surface in an isothermal family (except for the obvious case of a sphere).

In our preceding work,[2] we have obtained an extension of the theorem of Lie concerning isothermal families in the plane. We have found that a family of curves: $g(x, y)=\text{const.}$, on a surface Σ where any point is defined by general curvilinear coordinates (x, y), is isothermal if and only if the angle $\theta=\theta(x, y)$ between the family and the parametric curves $x=\text{const.}$ satisfies a certain partial differential equation of the second order in (x, y), and therefore if and only if $g(x, y)$ satisfies a certain partial differential equation of the third order in (x, y). Of course, if (x, y) are isothermal parameters on Σ, then our condition reduces to Lie's theorem which states that θ is a harmonic function of (x, y) if and only if the given family of curves is isothermal.

We have applied our result to the case in which the surface Σ is given by the Monge equation: $z=f(x, y)$, where (x, y, z) denote cartesian coordinates of space. The condition is

$$\begin{aligned}(1+f_y^2)\frac{\partial^2\theta}{\partial x^2} &- 2f_xf_y\frac{\partial^2\theta}{\partial x\partial y}+(1+f_x^2)\frac{\partial^2\theta}{\partial y^2}\\ &-(1+f_x^2+f_y^2)^{-1}[(1+f_y^2)f_{xx}-2f_xf_yf_{xy}\\ &+(1+f_x^2)f_{yy}]\left[f_x\frac{\partial\theta}{\partial x}+f_y\frac{\partial\theta}{\partial y}\right]+(1+f_x^2+f_y^2)^{1/2}\\ &\cdot\frac{\partial}{\partial y}\left\{\frac{f_x[(1+f_y^2)f_{xx}-2f_xf_yf_{xy}+(1+f_x^2)f_{yy}]}{(1+f_y^2)(1+f_x^2+f_y^2)}\right\}=0,\end{aligned}\tag{1}$$

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[1] The work of the present paper is concerned not only with real euclidean space but also with the complex euclidean space.

[2] Kasner and DeCicco, *An extension of Lie's theorem on isothermal families*, Proc. Nat. Acad. Sci. U.S.A. vol. 31 (1945) pp. 44–50.