ON THE EQUATIONS OF MOTION IN A RIEMANN SPACE

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Introduction. The object of this note is to give a geometric interpretation to the conditions of integrability of Killing's equations[1] and to the similar equations for collineations[2] in an affinely connected space. This is done by considering the more general problem of mapping a space upon itself, the mapping preserving some differential invariant. We shall be concerned with purely local properties so that the functions involved are assumed to have continuous derivatives of any necessary order and the groups are assumed to be infinitesimal.

1. **Infinitesimal mapping of V_n upon itself.** By an infinitesimal mapping of V_n upon itself we understand a correspondence generated by a vector field $\xi^i(x)$; that is, to a point $P(x)$ corresponds a point $P(\bar{x})$ where

$$\bar{x}^i = x^i + \xi^i(x)\delta t. \tag{1.1}$$

In order to see the effect of such a mapping on a differential invariant, we consider—merely to be specific—a tensor field of components $T^i{}_j(x)$. Such a tensor field is mapped into one whose components are

$$\overline{T}^i{}_j(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^\alpha}\cdot\frac{\partial x^\beta}{\partial \bar{x}^j} T^\alpha{}_\beta(x),$$

which for the mapping (1.1) gives

$$\overline{T}^i{}_j(\bar{x}) = T^i{}_j(x) + \left(\frac{\partial \xi^i}{\partial x^\alpha} T^\alpha{}_j - \frac{\partial \xi^\beta}{\partial x^j} T^i{}_\beta\right)\delta t. \tag{1.2}$$

By the variation of $T^i{}_j(x)$ we shall understand $\lim_{\delta t\to 0}(T^i{}_j(\bar{x}) - \overline{T}^i{}_j(\bar{x}))/\delta t$ and shall denote it by $\delta T^i{}_j(x)/\delta t$. Then (1.2) gives

$$\frac{\delta T^i{}_j(x)}{\delta t} = \xi^\alpha \frac{\partial T^i{}_j}{\partial x^\alpha} - \frac{\partial \xi^i}{\partial x^\alpha} T^\alpha{}_j + \frac{\partial \xi^\alpha}{\partial x^j} T^i{}_\alpha, \tag{1.3}$$

with a corresponding expression for any scalar or tensor field. If a tensor field is to remain invariant, it is necessary and sufficient that its variation be zero. From (1.3) it is obvious that a zero tensor field

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[1] L. P. Eisenhart, *Riemannian geometry*, chap. 6.

[2] L. P. Eisenhart, *Non-Riemannian geometry*, chap. 3; our $R^i{}_{jkl}$ is the negative of that used by Eisenhart.