METRIC PROPERTIES OF A CLASS OF QUADRATIC DIFFERENTIAL FORMS

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Introduction. In the present paper a new invariant quadratic differential form Ω is geometrically defined for a general pair of surfaces S, S' whose corresponding points x, x' determine the metric normal to S at x. The ratio of the form Ω to the first fundamental form ds^2 of S, in which Ω and ds^2 are defined for a common arc element of S at x, is found to be independent of the direction of the element if and only if the surface S' is the locus of the center of mean curvature of S; the ratio thus determined is the Gaussian curvature K of S at x. We introduce at a point x of S the concept of conjugate elements of a given arc element of a conjugate net and prove that the form Ω for an arbitrary arc element is identical with the form Kds^2 for either conjugate element if and only if the surface S' is the plane net at infinity. The principal directions at x of the tensor whose components are the coefficients of the form Ω are the classical principal directions of S at x for an arbitrary choice of S'. Finally, we characterize the net of lines of mean-curvature of S and the mean-conjugate net of S as integral nets of equations of the form $\Omega=0$, in which the forms Ω are defined with respect to certain geometrically determined transforms S' of S. The method of the present paper employs dual systems of linear homogeneous equations of the first order in compact forms which facilitate the use of a tensor notation with homogeneous cartesian point and plane coordinates.

1. **The fundamental differential equations.** The rectangular cartesian coordinates of a generic point x of an analytic surface S are defined by single-valued functions of two independent parameters u^1, u^2,

$$x^i = x^i(u^1, u^2), \qquad i = 0, 1, 2.$$

Let $g_{\alpha\beta}$ and $g^{\alpha\beta}$ denote the covariant and contravariant metric tensors of S, respectively, and let $d_{\alpha\beta}$ denote the second fundamental covariant tensor of S. It is known [1, p. 220][1] that the direction cosines z^i of the normal to S at x and the functions x^i are solutions of the differential equations

$$(1.1) \qquad \frac{\partial^2 x}{\partial u^\beta \partial u^\alpha} = \left\{ {\gamma \atop \alpha\beta} \right\} \frac{\partial x}{\partial u^\alpha} + d_{\alpha\beta} z, \quad \frac{\partial z}{\partial u^\alpha} = \binom{\beta}{3\alpha} \frac{\partial x}{\partial u^\beta}, \qquad \alpha, \beta, \gamma = 1, 2,$$

Received by the editors March 23, 1945.

[1] Numbers in brackets refer to the bibliography at the end of the paper.