ON SAINT VENANT'S PRINCIPLE

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The so-called principle of statically equivalent loads, due to Saint Venant, has been referred to for the last fifty years in almost all texts on elasticity. The statements in different books vary only slightly. Let us quote A. E. H. Love's *Treatise on the mathematical theory of elasticity* (4th ed., p. 132): "According to this principle the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part." In this form the statement is not very clear. Forces applied to a body at rest must be in equilibrium in any case. It would not make sense to speak of adding or subtracting a system of forces that is not an equilibrium system. What is meant may be correctly expressed in this way: If the forces acting upon a body are restricted to several small parts of the surface, each included in a sphere of radius ϵ, then the strains and stresses produced in the interior of the body at a finite distance from all those parts are smaller in order of magnitude when the forces for each single part are in equilibrium than when they are not. If this statement is true, it must be capable of a mathematical proof, that is, it must be a consequence of the fundamental differential equations of elasticity theory. But no attempt is made in the usual textbooks to supply a demonstration. Most texts give Boussinesq as a reference for the proof. What Boussinesq really dealt with was the infinite body filling the half space $z>0$ and subjected to normal forces at its boundary $z=0$. If the forces are applied to points ξ, η, 0 where $\xi^2+\eta^2\leqq\epsilon^2$, Boussinesq proved that the stress at a point x, y, z is of order ϵ when the sum of forces is zero and of order ϵ^2 when their moments also vanish. It will be shown in the following that this is not the case, in general, if tangential components of the forces at $z=0$ are admitted. Moreover we shall consider a body of finite dimensions and see that there too Saint Venant's principle in its traditional form does not hold true. The main result, from a practical point of view, is that Saint Venant's principle can be applied if all forces involved are parallel and not tangential to the surface of the body, but not under more general conditions. No objection is raised in the present paper against using the principle in the case of bodies with one or two infinitesimal dimensions, like thin plates, shells or beams, although a proof of its

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