A NOTE ON GROUPS WITHOUT ISOMORPHIC SUBGROUPS

IRVING KAPLANSKY

1. Introduction. One of the theorems obtained by R. A. Beaumont in a recent paper[1] states that if G is an abelian group of finite rank all of whose elements have finite order, then G has no proper isomorphic subgroups. If we interpret G as a vector space over the ring of integers I, it is natural to raise the question: what properties of I are needed for this result? In this note we find two conditions which are sufficient, given as (1) and (2) in our main theorem. Whether these conditions are also necessary remains to be determined.

2. A preliminary lemma. We shall require the following lemma, which is closely related to various known results.[2]

LEMMA. *Let G be a vector space over a commutative principal ideal ring R. Suppose that G can be spanned by r elements, and that H is a subspace that can be spanned by a finite number of elements of H. Then H can be spanned by r elements of H.*

PROOF. Let G be spanned by $g_1, \cdots, g_r$; H by $h_1, \cdots, h_s$. Then $h_i = \sum_j \alpha_{ij} g_j$ $(\alpha_{ij} \in R)$. Let β be the H.C.F. of $\alpha_{11}, \cdots, \alpha_{s1}$, so that $\beta = \sum_i \lambda_i \alpha_{i1}$. Define $k_1 = \sum_i \lambda_i h_i$. Then $k_1 \in H$, and since

$$k_1 = \beta g_1 + \text{a linear combination of } g_2, \cdots, g_r,$$

$k_1, g_2, \cdots, g_r$ span H. After r such steps we shall obtain elements $k_1, \cdots, k_r$ in H which span H.

3. The theorem. Let V be a vector space over a ring R.

DEFINITION. V has rank r over R if any finite subset of V can be spanned over R by r elements of V, and if r is the smallest integer with this property.

THEOREM. *Let V have finite rank r over R and suppose that:*

(1) *R is a commutative principal ideal ring.*

(2) *Every proper residue class ring of R is finite.*

(3) *For every $v \in V$, there is an $\alpha \neq 0$ in R such that $\alpha v = 0$.*

Then V has no isomorphic proper subspaces.

Presented to the Society, April 28, 1945; received by the editors March 12, 1945.

[1] *Groups with isomorphic proper subgroups*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 381–387.

[2] For example, C. J. Everett, *Vector spaces over rings*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 312–316.