A NEW SOLUTION FOR LINEAR DIFFERENCE EQUATIONS

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We shall give a solution of the fundamental difference equation:

$$F(t+1) - F(t) = \phi(t). \tag{1}$$

As to our method, we stress that the theory of Fourier series is used. Accordingly, the variable t is assumed to be real. We suppose that $\phi(t)$ is integrable and has bounded variation in every finite interval of t or satisfies any other condition sufficient for expansion in a Fourier series. For simplicity we at first assume that $\phi(t)$ is continuous. Our solution is

$$F(t) = -\frac{\phi(t)}{2} + \int_a^t \phi(\tau)d\tau + 2\sum_{k=1}^{\infty}\int_a^t \phi(\tau)\cos 2\pi k(t-\tau)d\tau, \tag{2}$$

where a is constant.

The above series is not a Fourier series in the usual sense, since the upper limit of the integrals is not constant, but the variable t itself.

To prove the truth of our statement we compute the difference

$$\begin{aligned} F(t+1) - F(t) = &-\frac{\phi(t+1)-\phi(t)}{2} + \int_t^{t+1}\phi(\tau)d\tau \\ &+ 2\sum_{k=1}^{\infty}\int_t^{t+1}\phi(\tau)\cos 2\pi k(t-\tau)d\tau. \end{aligned} \tag{3}$$

Now, expansion of $\phi(t)$ in a Fourier series in any interval of length 1, say in the interval $c \cdots c+1$, c meaning an arbitrary real constant, gives

$$\phi(t) = \int_c^{c+1}\phi(\tau)d\tau + 2\sum_{k=1}^{\infty}\int_c^{c+1}\phi(\tau)\cos 2\pi k(t-\tau)d\tau.$$

The series represents $\phi(t)$ in the interior of the said interval, but it represents the value $(\phi(c)+\phi(c+1))/2$ at either end point, say at the left end point c. Therefore

$$\frac{\phi(c)+\phi(c+1)}{2} = \int_c^{c+1}\phi(\tau)d\tau + 2\sum_{k=1}^{\infty}\int_c^{c+1}\phi(\tau)\cos 2\pi k(c-\tau)d\tau.$$

This holds for every c and we can write t instead of c, giving

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