THE ASYMPTOTIC DEVELOPMENTS OF A CLASS OF ENTIRE FUNCTIONS

H. K. HUGHES

1. Introduction. In an earlier paper [2],[1] the author established a general theorem concerning the asymptotic behavior in the neighborhood of the point at infinity of an entire function $f(z)$ defined by a Maclaurin series of the form

$$\sum_{n=0}^{\infty} g(n)z^n, \tag{1}$$

where the coefficient $g(n)$ of z^n satisfies certain conditions. In the present paper, we wish to show an application of this theorem. We shall use it to find the asymptotic expansions of an important class of entire functions, namely those defined by the series

$$\sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha_1 n + s_1)\Gamma(\alpha_2 n + s_2) \cdots \Gamma(\alpha_m n + s_m)} \qquad (m \geqq 2), \tag{2}$$

where $\alpha_j > 0$ $(j=1, 2, \cdots, m)$ and s_j are any constants, real or complex. Functions of this type are frequently connected with the solutions of linear differential equations. Moreover, a special case of (2) is the so-called generalized Bessel function, namely

$$\phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(ln + p)} \qquad (l > 0),$$

which has important applications in partition theory.

Throughout the paper, we shall denote by $f(z)$ the function defined by series (2), and by $g(n)$ the coefficient of z^n in (2). Moreover, we shall adopt single symbols to represent certain combinations of the quantities m, α_j, and s_j, these being as follows:[2]

$$\begin{gathered} \alpha = \sum_{j=1}^{m} \alpha_j; \qquad s = \sum_{j=1}^{m} s_j; \qquad \sigma = \alpha^{\alpha} \prod_{j=1}^{m} (\alpha_j)^{-\alpha(j)}; \\ \gamma = \prod_{j=1}^{m} (\alpha_j)^{s(j)-1/2}; \qquad t = s + \frac{1}{2} - \frac{1}{2} m; \qquad c = \frac{\alpha^{t-3/2}}{(2\pi)^{(m-1)/2} \cdot \gamma}. \end{gathered} \tag{3}$$

Presented to the Society, November 25, 1944; received by the editors December 29, 1944.

[1] Numbers in square brackets apply to the references at the end of the paper.

[2] We use the symbol $\alpha(j)$ to denote α_j when the latter occurs as an exponent. The symbols $s(j)$ in (3) and $\beta(j)$, and so on, in (6) have a similar meaning.