A NOTE ON KLOOSTERMAN SUMS

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1. **Introduction.** In recent years the Kloosterman sum

$$A_k(n) = \sum_{h \bmod k}{}' \exp\left(2\pi i n(h + \bar{h})/k\right)$$

has played an increasingly important role in the analytic theory of numbers. The dash $'$ beside the summation symbol indicates that the letter of summation runs only through a reduced residue system with respect to the modulus. The number $\bar{h}$ is defined as any solution of the congruence $h\bar{h} \equiv 1 \pmod{k}$, and n denotes an arbitrary integer. It was shown by Salié[1] almost fifteen years ago that $A_k(n)$ may be evaluated explicitly when k is a power of a prime. Salié's result is given by the following theorem.

THEOREM. *Let $k = p^\alpha$, $\alpha \geqq 2$, $(n, k) = 1$, where p denotes an odd prime. Then,*

(i) *if α is even,*

$$A_k(n) = 2k^{1/2} \cos(4\pi n/k);$$

(ii) *if α is odd,*

$$A_k(n) = \begin{cases} 2(n \mid k)k^{1/2} \cos(4\pi n/k) & \text{for } p \equiv 1 \pmod{4}, \\ -2(n \mid k)k^{1/2} \sin(4\pi n/k) & \text{for } p \equiv 3 \pmod{4}. \end{cases}$$

The symbol $(n \mid k)$ denotes, as is usual, the Legendre symbol.

Salié's proof of his theorem is based upon induction. In the present note a direct proof is given. The method consists in introducing a transformation which expresses the Kloosterman sum in terms of Gauss sums and certain types of Ramanujan sums.

2. **Two lemmas.** A Gauss sum may be defined by

$$G_{h,k} = \sum_{m=0}^{k-1} \exp(2\pi i h m^2/k).$$

We shall find it convenient to write G instead of $G_{1,k}$. The following lemma[2] is classical.

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[1] Hans Salié, *Über die Kloostermanschen Summen* $S(u, v; q)$, Math. Zeit. vol. 34 (1931) pp. 91–109.

[2] See, for example, Edmund Landau, *Vorlesungen über Zahlentheorie*, vol. 1, p. 153.