PARALLELISM IN THE TENSOR ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

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In the extensions of classical differential geometry associated with ordinary differential equations the connection coefficients may be chosen arbitrarily, subject to certain laws of transformation under change of coordinates. But the calculations of differential invariants become decidedly simpler if the choice is made in a particular way, so as to reduce the number of "invariants of the connection" to a minimum, leaving only the intrinsic invariants of the system. For partial differential equations, no such intrinsic choice had hitherto been found, though something of the sort is badly needed inasmuch as the mass of calculations is far more complicated than for any system of ordinary differential equations.

Given the system of partial differential equations

$$
\begin{gathered}
\frac{\partial^{q+1} x^i}{\partial u^{\alpha_1} \cdots \partial u^{\alpha_{q+1}}} + \overset{i}{H}_{\alpha_1 \cdots \alpha_{q+1}}(u, x, \overset{i}{p}_{\alpha_1} \cdots \overset{i}{p}_{\alpha_1, \cdots, \alpha_q}) = 0; \\
\overset{i}{p}_\alpha = \frac{\partial \overset{i}{x}}{\partial u^\alpha}, \cdots, \overset{i}{p}_{\alpha_1 \cdots \alpha_q} = \frac{\partial \overset{i}{p}_{\alpha_1 \cdots \alpha_{q-1}}}{\partial u^{\alpha_q}}, \\
i, j, k, \cdots = 1, 2, \cdots, n; \alpha, \beta, \gamma, \cdots = 1, 2, \cdots, m,
\end{gathered} \tag{1}
$$

the complete set of tensor operators and tensor differential invariants associated with the "path-space" of this system may be built up as follows. We assume that (a) the equations (1) have a manifold of solutions; (b) they are tensorial under nonsingular point-transformations in both the x and the u coordinates; (c) the equations both of the x-variation and the u-variation of (1) are tensor-invariant and (d) there exists a tensorial differentiation operator

$$
D_\alpha T^{\cdots}_{\cdots} = \partial_\alpha T^{\cdots}_{\cdots} + \overset{i}{\gamma}_{\alpha r} \overset{r \cdot \cdot}{T}_{\cdots} - \overset{r}{\gamma}_{\alpha j} T^{\cdots}_{r \cdot \cdot} + \Gamma^\nu_{\alpha\rho} T^{\cdot \cdot \rho}_{\cdots} - \Gamma^\rho_{\alpha\sigma} T^{\cdots}_{\cdot \cdot \rho} \tag{2}
$$

(one summation for every index of the tensor $T^{\cdots}_{\cdots}$), where

$$
\partial_\alpha = \frac{\partial}{\partial u^\alpha} + \overset{i}{p}_\alpha \frac{\partial}{\partial x^i} + \overset{i}{p}_{\alpha\beta} \frac{\partial}{\partial \overset{i}{p}_\beta} + \cdots - \overset{i}{H}_{\alpha\beta_1 \cdots \beta_q} \frac{\partial}{\partial \overset{i}{p}_{\beta_1 \cdots \beta_q}} \cdot
$$

For the tensor invariance of this operator, it is necessary and sufficient that the connection coefficients $\gamma^i_{\alpha j}$, $\Gamma^\alpha_{\beta\nu}$ obey the transformation laws:

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