ON THE DEGREE OF APPROXIMATION OF FUNCTIONS BY FEJÉR MEANS

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1. Continuous functions. It has been proved by S. Bernstein that if $f(x)$ is periodic and of the class Lip α, $0<\alpha<1$, then the $(C, 1)$ means $\sigma_n(x)=\sigma_n(x;f)$ of the Fourier series of f satisfy the condition

$$\sigma_n(x) - f(x) = O(n^{-\alpha}), \tag{1.1}$$

uniformly in x. The result is false for $\alpha=1$. The place of (1.1) is then taken by

$$\sigma_n(x) - f(x) = O(\log n/n), \tag{1.2}$$

and, as simple examples show, the factor $\log n$ on the right cannot be removed (see, for example, A. Zygmund, *Trigonometrical series*, p. 62). It will be shown here that for power series the inequality (1.1) holds even for $\alpha=1$. More generally, we have the following theorem.

THEOREM 1. *Suppose that $f(x)$ is periodic, continuous, and that the Fourier series of f is of power series type,*

$$f(x) \sim \sum_{\nu=0}^{\infty} c_\nu e^{i\nu x}.$$

Then

$$| \sigma_{n-1}(x) - f(x) | \leqq A\omega(2\pi/n), \tag{1.3}$$

where $\omega(\delta)$ is the modulus of continuity of f and A is an absolute constant.

The proof is based on the following lemma.

LEMMA. *Suppose that*

$$g(x) \sim \sum_{-\infty}^{+\infty} \gamma_\nu e^{i\nu x} \tag{1.4}$$

satisfies $|g(x+h)-g(x)| \leqq M|h|$. Then

$$| \tilde{\sigma}_{n-1}(x) - \tilde{g}(x) | \leqq BM/n, \tag{1.5}$$

where $\tilde{g}(x)$ is the function conjugate to $g(x)$ and $\tilde{\sigma}_n(x)$ are the $(C, 1)$ means of the series conjugate to (1.4).

For the proof of the lemma we note that

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