NOTES ON LEGENDRE POLYNOMIALS

EARL D. RAINVILLE

1. Introduction. We shall obtain three results. The first is a generating function for the Legendre polynomials that lies between the classical

$$(1 - 2xW + W^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)W^n$$

and a special case of a well known result of Bateman.[1] The second is an expression for $P_n(\cos \alpha)$ as a series in $P_k(\cos \beta)$ with α and β unrelated. Special cases of the latter are known. The third result is a relation in integral form between the Hermite and Legendre polynomials.

2. A generating function for $P_n(x)$. From Laplace's first integral,

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} [x + (x^2 - 1)^{1/2} \cos \beta]^n d\beta,$$

we find by direct expansion and the use of Wallis' formula the known[2] result

$$P_n(x) = n! \sum_{k=0}^{[n/2]} (-1)^k \frac{(1 - x^2)^k x^{n-2k}}{2^{2k}(k!)^2(n - 2k)!}, \tag{1}$$

where [ ] is the greatest integer symbol.

Examination of (1) in the light of the identity

$$\left(\sum_{n=0}^{\infty} a_n y^{2n}\right)\left(\sum_{n=0}^{\infty} b_n y^n\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} a_k b_{n-2k} y^n, \tag{2}$$

and of the power series for the Bessel function $J_0(y)$,

$$J_0(y) = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{2^{2n}(n!)^2},$$

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[1] H. Bateman, *A generalization of the Legendre polynomial*, Proc. London Math. Soc. (2) vol. 3 (1905) pp. 111–123. I am much indebted to Professor Bateman for pointing out that the generating function in (3) may be found in the work of Catalan, F. H. Jackson, and others. This relation seems to have been less widely used than it deserves. The short, possibly new, derivation in §2 may contribute to an understanding of the material in §§3 and 4.

[2] E. W. Barnes, *On generalized Legendre functions*, Quarterly Journal of Pure and Applied Mathematics vol. 39 (1908) pp. 97–204. See p. 120.