EXTENSION OF A THEOREM OF BOCHNER ON EXPRESSING FUNCTIONALS AS RIEMANN INTEGRALS

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Introduction. S. Bochner[1] has shown that an additive homogeneous functional defined over a sufficiently large class C of functions can be realized as a Riemann integral with respect to a finitely additive measure V in the space X over which the functions are defined. His proof makes use of the fact that the constant function belongs to C, as a result, $V(X)$ is finite. It is the purpose of this note to show that a similar theorem holds even when $V(X)$ turns out to be infinite. A modification of Bochner's proof would suffice for this stronger theorem. We have chosen rather to treat it as a problem of extending the domain of definition of the given functional.

Throughout we have used the symbol $\rightarrow$ to be read as "implies." The equality $\equiv$ is used to denote an equality which holds by definition.

Notations. We consider a space X of points x, and real-valued point functions $f, g, \cdots$ over X. Given f, g, and real numbers a, b, we shall write

$$|f|,\ af+bg,\ fg,\ f\wedge g,\ f\vee g,\ f^+,\ f^-,$$

respectively, for those functions whose values for each x are given by

$$|f(x)|, \quad af(x)+bg(x), \quad f(x)g(x), \quad \inf\,[f(x), g(x)],$$
$$\sup\,[f(x), g(x)], \quad \sup\,[f(x), 0], \quad \sup\,[-f(x), 0].$$

We shall write a for the constant function $f(x)=a$, and write $f\geqq g$ if for each x, $f(x)\geqq g(x)$. The function which coincides with f on a set A and is equal to 0 in $X-A$ will be denoted by f_A. In particular we write 1_A for the characteristic function of the set A. The symbol $\emptyset$ will denote the empty set.

It is clear that $f=f^+-f^-$, and that

$$(f_A)^+ = (f^+)_A, \qquad (f_A)^- = (f^-)_A.$$

1. *R*-**measure.**

1.1. By an R-measure in X we shall mean a set function $V(E)$ defined for sets E of a family $\mathbf{A}$ with the following properties:

Received by the editors January 10, 1943, and, in revised form, September 5, 1944.

[1] S. Bochner, *Additive set functions on groups*, Ann. of Math. vol. 40 (1939) pp. 769–799. The theorem in question occurs in paragraph 4.