THE EQUATION $x' \equiv xd - dx = b$

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Let $\mathfrak{A}$ be an associative algebra with a possibly infinite basis over a field Φ. Then if d is a fixed element in $\mathfrak{A}$, it is well known that the mapping $x \rightarrow x' \equiv [x, d] = xd - dx$ is a derivation[1] in $\mathfrak{A}$; that is,

$$(x + y)' = x' + y', \qquad (x\alpha)' = x'\alpha, \qquad (xy)' = x'y + xy'$$

for all x, y in $\mathfrak{A}$ and all α in Φ. The constants relative to such a derivation are the elements of $\mathfrak{A}$ that commute with d. We shall call an element b a *d-integral* if $b = a'$ for some element a in $\mathfrak{A}$, that is, if the equation $x' = xd - dx = b$ has a solution in $\mathfrak{A}$. Clearly if a is a solution of this equation then the totality of solutions is the set $\{a + c\}$ where c ranges over the set of d-constants. In a recent paper appearing in this Bulletin, R. E. Johnson obtained a necessary and sufficient condition that an element b be a d-integral under the assumption that $\mathfrak{A}$ is a separable algebraic division ring.[2] In this note we allow $\mathfrak{A}$ to be an arbitrary algebra but we make the assumption that d is an algebraic element in the sense that it satisfies a polynomial equation with coefficients in Φ. We obtain a necessary condition, which is equivalent to Johnson's condition when $\mathfrak{A}$ is a division ring, that b be a d-integral. If the minimum polynomial $\mu(\lambda)$ of d is relatively prime to its derivative $\mu'(\lambda)$, then it is easy to see that the condition is also sufficient and one may give an explicit formula for a solution of the equation $x' = b$. If we assume that $\mathfrak{A}$ is a simple algebra satisfying the descending chain condition for left ideals then we can show that our condition is also sufficient when $\mu(\lambda)$ is a product of distinct irreducible factors in $\Phi[\lambda]$ and in certain other cases. Here, however, we do not display a solution but merely prove its existence. Our results include, of course, Johnson's result for algebraic division rings, since the minimum polynomial of an element in such a ring is irreducible. No assumption about separability is required.

In order to obtain a condition for the solvability of the equation $x' = b$ we consider the matrices

$$(1) \qquad u = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}, \qquad v = \begin{pmatrix} d & b \\ 0 & d \end{pmatrix}$$

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[1] Cf. the author's paper *Abstract derivation and Lie algebras*, Trans. Amer. Math. Soc. vol. 42 (1937) pp. 206–224.

[2] *On the equation* $\chi\alpha = \gamma\chi + \beta$ *over an algebraic division ring*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 202–208.