A GENERALIZATION OF CONTINUED FRACTIONS[1]

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1. **Introduction.**[2] The generalizations and analogues of regular continued fractions due to Pierce [8], Lehmer [5], and Leighton [6] concern the iteration of rational functions to obtain rational approximations to a real number. The present generalization proceeds from the fact that the continued fraction

$$\frac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} \tag{1.1}$$

can be written in the form

$$f(a_1 + f(a_2 + \cdots \tag{1.2}$$

where $f(t) = 1/t$. This suggests the possibility of using functions other than $1/t$ to obtain generalizations of (1.1). In §2 a class F of functions which includes $1/t$ is defined and in §3 meaning is given to (1.2) for each $f \in F$ and each sequence $a_1, a_2, a_3, \cdots$ of positive integers. An algorithm is given for obtaining for a fixed $f \in F$ an expression of the form (1.2) corresponding to each number x in the interval $0 < x < 1$; this expression is then called the *f-expansion of x*. The analogue of the nth convergent of a simple continued fraction is defined, and its behavior with respect to x is noted. In §4 the form (1.2) is called an *f-expansion* when $f \in F$ and $a_1, a_2, a_3, \cdots$ is a sequence of positive integers. The convergence and some idea of the rapidity of convergence of an f-expansion are established. The one-to-one correspondence between f-expansions and f-expansions of numbers x, $0 < x < 1$, is given in §5 by Theorem 5. In §6 statistical independence of the a_i of an f-expansion is defined in the customary way and a subclass F_p of F for which the a_i are statistically independent is considered. Various sets of numbers x whose f-expansions are restricted by conditions on the a_i are considered and the linear Lebesgue measures of these sets are given. In §7, when $f \in F_p$, certain sets of numbers x which have been studied for $f(t) = 1/t$ by Borel [2] and F. Bernstein [1] are shown to be of measure zero.

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[2] Numbers in brackets refer to the bibliography.