HADAMARD'S THREE CIRCLES THEOREM

RAPHAEL M. ROBINSON

Hadamard's theorem is concerned with the relation between the maximum absolute values of an analytic function on three concentric circles.[1] If we put

$$M(r) = \max_{|z|=r} |f(z)|,$$

then the theorem states that $\log M(r)$ is a convex function of $\log r$ for $r'<r<r''$, if $f(z)$ is regular for $r'<|z|<r''$. This is an immediate consequence of the fact that if $|f(z)| \leqq A|z|^\lambda$ on two circles about the origin, then it is also true between the circles; and this in turn is seen by applying the principle of maximum to $f(z)/z^\lambda$. The bound is attainable within the ring only for $f(z)=\alpha z^\lambda$ with $|\alpha|=A$. Notice that this function is single-valued only if λ is an integer, so that Hadamard's bound is not in general sharp for single-valued functions. (It is the sharp bound for the class of many-valued functions, any branch of which is regular in the ring, and for which $|f(z)|$ is single-valued.)

We shall consider only single-valued functions. The problem of finding the sharp bound in Hadamard's theorem is formulated as Problem A below. (It is no essential restriction to suppose that the radius of the outer circle is 1, and that the given bound on this circle is 1.) Problems B and C raise the same question for more special classes of functions.

PROBLEM A. *Suppose* $0<q<Q<1$ *and* $p>0$. *Consider the class of functions satisfying the following conditions*: $f(z)$ *is regular for* $q \leqq |z| \leqq 1$,

$$|f(z)| \leqq 1 \quad \textit{for} \quad |z| = 1, \qquad |f(z)| \leqq p \quad \textit{for} \quad |z| = q.$$

An address delivered before the Berkeley meeting of the Society on April 29, 1944, by invitation of the Program Committee; received by the editors May 6, 1944.

[1] The theorem was stated (without proof) in Hadamard's note, *Sur les fonctions entières*, Bull. Soc. Math. France vol. 24 (1896) pp. 186–187. His proof was apparently first published in 1912; it may be found in footnote 2, p. 94, of *Selecta: Jubilé Scientifique de M. Jacques Hadamard*, Paris, 1935. In the meantime, proofs (of a less simple nature) had been given by O. Blumenthal and by G. Faber. See Blumenthal, *Über ganze transzendente Funktionen*, Jber. Deutschen Math. Verein. vol. 16 (1907) pp. 97–109, and *Sur le mode de croissance des fonctions entières*, Bull. Soc. Math. France vol. 35 (1907) pp. 213–232; Faber, *Über das Anwachsen analytischer Funktionen*, Math. Ann. vol. 63 (1907) pp. 549–551.