SIMPLE QUASIGROUPS

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Introduction. A. A. Albert [1, II][1] has conjectured that *there exist simple loops of every finite order except order* 4. This conjecture is established in §1 by the construction of what we call *hyperabelian loops*.[2] In §2 other simple loops are constructed, in particular, loops of order $2f+1$ with subloops of order f. The concluding section of the paper is devoted to an investigation of the relationship between non-simple quasigroups (of finite or infinite order) and the loops isotopic to them (Theorem V). Theorems I and V of the paper have elsewhere been announced without proof by the author [2, §10]. For the purposes of §§1 and 2 none of the refinements of the extension theory for loops and quasigroups will be needed [1, 2, 4] and we shall be content in this introductory section with a few remarks sufficient for the proof of Lemma 1 below.

A non-empty set Q of elements $a, b, \cdots$ is said to be a *quasigroup* if (I) *to every ordered pair* $a, b \subset Q$ *there corresponds an element* $ab = c \subset Q$ and (II) *when any two of the symbols* x, y, z *of the equation* $xy = z$ *are assigned as elements of* Q *the third is uniquely determined as an element of* Q. In particular a quasigroup is called a *loop* if it possesses a (unique) unit element. Obviously every group is a loop. The *order* of a quasigroup is, by definition, its cardinal number, finite or transfinite.

If a quasigroup Q is homomorphic to a quasigroup R we may speak of R as a *proper homomorph* of Q if (i) *R is not isomorphic to Q*, (ii) *R is not a group of order one*. Correspondingly a quasigroup is *simple* if it has no proper homomorphs. This definition of simplicity is equivalent to the usual one in the case of groups, as well as to that used for finite quasigroups by G. N. Garrison [4] and to that employed by Albert [1, II] for arbitrary loops.

If a non-simple quasigroup Q has a proper homomorph R we may designate by H_p the set of elements of Q which map into the element p of R under a given homomorphism of Q upon R. Obviously H_p and H_q have common elements if and only if $p = q$. If $pq = r$, let x, y, z designate arbitrary elements of the set H_p, H_q, H_r respectively. Then,

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] The name hyperabelian loop was suggested by the fact that the multiplication table for such a loop F_G is built upon that of a commutative (or abelian) loop. When G is a cyclic group we call F_G hypercyclic.