SUBDIRECT UNIONS IN UNIVERSAL ALGEBRA

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1. **Preliminary definitions.** By an *algebra*, we shall mean below any collection A of elements, combined by any set of single-valued operations f_α,

$$y = f_\alpha(x_1, \cdots, x_{n(\alpha)}). \tag{1}$$

The number of distinct operations (that is, the range of the variable α) may be infinite, but for our main result (Theorem 2), we shall require every $n(\alpha)$ to be finite—that is, it will concern *algebras with finitary operations*.

The concepts of *subalgebra*, *congruence relation* on an algebra, *homomorphism* of one algebra A onto (or into) another algebra with the same operations, and of the *direct union* $A_1 \times \cdots \times A_r$ of any finite or infinite class of algebras with the same operations have been developed elsewhere.[1] More or less trivial arguments establish a many-one correspondence between the congruence relations θ_i on an algebra A and the homomorphic images $H_i = \theta_i(A)$ of the algebra (isomorphic images being identified); moreover the congruence relations on A form a lattice (the *structure lattice* of A). In this lattice, the equality relation will be denoted 0; all other congruence relations will be called *proper*.

More or less trivial arguments also show (cf. *Lattice theory*, Theorem 3.20) that the isomorphic representations of any algebra A as a *subdirect union*, or subalgebra $S \leqq H_1 \times \cdots \times H_r$ of a direct union of algebras H_i, correspond essentially one-one to the sets of congruence relations θ_i on A such that $\wedge \theta_i = 0$. In fact, given such a set of θ_i, the correspondence

$$\theta: \quad a \rightarrow [\theta_1(a), \cdots, \theta_r(a)] = [h_1, \cdots, h_r] \tag{2}$$

exhibits the desired isomorphism of A with a subalgebra of $H_1 \times \cdots \times H_r$, where $H_i = \theta_i(A)$. Incidentally, the number of S_i can

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[1] *On the structure of abstract algebras*, Proc. Cambridge Philos. Soc. vol. 31 (1935) pp. 433–454, and in the foreword to the author's *Lattice theory*. The idea of an abstract congruence relation is also developed in chap. VI, §14, of S. MacLane's and the author's *Survey of modern algebra*. Interesting remarks in this connection may be found in J. C. C. McKinsey's and A. Tarski's *The algebra of topology*, Ann. of Math. vol. 45 (1944) esp. pp. 190–191.