INTEGRAL THEOREMS IN THREE-DIMENSIONAL POTENTIAL FLOW

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1. Introduction. A potential flow can be described as a vector distribution $\bar{q}(\bar{r})$ ($\bar{q}$ = velocity vector, $\bar{r}$ = position vector) subject to the two conditions

$$\text{div } \bar{q} = 0, \qquad \text{curl } \bar{q} = 0. \tag{1}$$

Instead of (1) one can also ask that a scalar function $\phi(\bar{r})$ exist so that

$$\bar{q} = \text{grad } \phi, \qquad \Delta\phi = 0. \tag{2}$$

If both $\bar{q}$ and $\bar{r}$ are restricted to two dimensions, a third form of representation is possible. One can combine the components x, y of $\bar{r}$ and u, v of $\bar{q}$ to two complex numbers

$$x + iy = \zeta, \qquad u - iv = \mathfrak{v} \tag{3}$$

and then state that $\mathfrak{v}$ is an analytic function of ζ. In this case, the Cauchy formula holds,

$$\int f(\mathfrak{v}, \zeta) d\zeta = 0, \tag{4}$$

if f is an analytic function of $\mathfrak{v}$ and ζ and the integral is extended over the complete boundary of a region in which f is regular.

In the dynamics of the two-dimensional potential flow several equations of the form (4) play a decisive role. It must be expected that the analogous theorems are valid in three-dimensional potential flow also. But the question has not yet been answered: *For what vector functions $\bar{f}$ of $\bar{q}$ and $\bar{r}$ is the equation*

$$\int \bar{f}(\bar{q}, \bar{r}) \cdot d\bar{S} = 0 \tag{5}$$

correct if the integral is extended over the complete boundary of a region in which $\bar{f}$ has continuous derivatives of the first order with respect to the six components x, y, z of $\bar{r}$ and u, v, w of $\bar{q}$? Here, obviously, $d\bar{S}$ is the vectorial area element whose direction is that of the outward normal, and the dot means scalar multiplication. The surface may consist of a finite number of analytic pieces.

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