SUMMABILITY OF SUBSEQUENCES

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1. **Introduction.** Let a_{nk} $(n, k=1, 2, \cdots)$ be a matrix of real or complex constants for which

$$(1.1) \qquad \lim_{n\to\infty} a_{nk} = 0, \qquad k = 1, 2, 3, \cdots,$$

$$(1.2) \qquad \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1; \qquad \sum_{k=1}^{\infty} | a_{nk} | < M, \qquad n = 1, 2, 3, \cdots,$$

M being a constant. This matrix defines a regular method of summability by means of which a sequence x_n of real or complex numbers is summable to X if $X_n=\sum_{k=1}^{\infty}a_{nk}x_k$, $n=1, 2, 3, \cdots$, exists and $\lim X_n=X$. It has recently been shown by R. C. Buck[1] that if the sequence x_n is real, bounded, and divergent, then the sequence has a subsequence not summable A. This note proves the following more general theorem.

THEOREM. *Let A be regular and let x_n be a bounded complex sequence. Then there exists a subsequence y_n of x_n such that the set L_Y of limit points of the transform Y_n of y_n includes the set L_x of limit points of the sequence x_n.*

If x_n is a bounded divergent sequence, then L_x and hence also L_Y must contain at least two distinct points and accordingly the subsequence y_n is not summable A. Applying the theorem to the divergent sequence 0, 1, 0, 1, $\cdots$, we obtain the result of Steinhaus[2] that there is a sequence of 0's and 1's not summable A.

2. **Proof of the theorem.** Let L_x be the set of limit points of the bounded complex sequence x_n. Since the complex plane is separable and L_x is a closed set, there is a countable (finite or infinite) subset E of L_x such that the closure $\bar{E}$ of E is the set L_x itself. Let $u_1, u_2, u_3, \cdots$ be a sequence containing all of the points of E; in case E is a finite set, the points $u_1, u_2, u_3, \cdots$ are not distinct. Let the elements of the sequence

$$(2.1) \qquad u_1;\ u_1, u_2;\ u_1, u_2, u_3;\ \cdots;\ u_1, u_2, \cdots, u_n;\ \cdots$$

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[1] R. C. Buck, *A note on subsequences*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 898–899.

[2] H. Steinhaus, *Some remarks on the generalization of limit* (in Polish), Prace Matematyczno-fizyczne vol. 22 (1911) pp. 121–134.