THE MEAN CONVERGENCE OF NON-HARMONIC SERIES

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The theory of non-harmonic series deals with the completeness and expansion properties of sets of functions $\{e^{i\lambda_n x}\}$, $-\infty < n < \infty$. Extending results of Paley and Wiener, Levinson has obtained a theory of these functions in the spaces L^p, $1 < p \leqq 2$ [5, p. 113; 4, p. 48].[1] His principal result states that if

$$|\lambda_n - n| \leqq \delta \qquad (n = 0, \pm 1, \cdots) \tag{1}$$

with $\delta < (p-1)/2p$, λ_n real, then any function $f(x) \in L^p$ has a Fourier expansion in the above functions equiconvergent with its ordinary Fourier series in any closed interval $(-\pi+\epsilon, \pi-\epsilon)$.

This, however, leaves open the question whether the expansion converges to $f(x)$ in the topology of $L^p(-\pi, \pi)$. This is known to be true for all $p>1$ if $\lambda_n = n$ [6, p. 153]. For the non-harmonic case the only available results concern $p=2$: by refining an earlier result of Paley and Wiener, Duffin and Eachus [2, p. 855] obtain mean convergence for $\delta < \pi^{-1}\log 2$. (This is less than the $\delta < 1/4$ to be hoped for from Levinson's result.)

In this paper we shall generalize the Duffin-Eachus result to L^p, $p>1$. Our hypothesis is, however, more stringent than (1). We require that

$$\left(\sum_{-\infty}^{\infty} |\lambda_n - n|^a\right)^{1/a} = \delta < \pi^{-1}\log 2, \qquad a = \frac{2p}{|2-p|}. \tag{2}$$

If $p \to 2$ this becomes $\sup |\lambda_n - n| < \pi^{-1}\log 2$, which is the result we are extending.

It is interesting to observe that (2) remains the same if we replace p by its conjugate p', where $1/p + 1/p' = 1$. For then $|2p/(p-2)| = |2p'/(2-p')|$. We note also that the λ_n need not be real.

THEOREM. *Let $\{\lambda_n\}$ satisfy (2). Then any function $f(x) \in L^p$, $p>1$, admits a unique representation*

$$f(x) = \underset{N\to\infty}{\overset{(p)}{\text{l.i.m.}}} \sum_{-N}^{N} c_n e^{i\lambda_n x}. \tag{3}$$

Moreover

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[1] Numbers in brackets refer to the references listed at the end of the paper.