PROOF OF A CONJECTURE OF P. ERDÖS ON THE DERIVATIVE OF A POLYNOMIAL

PETER D. LAX

Introduction. We start out from the following consequence of S. Bernstein's well known theorem on trigonometric polynomials. Let $p_n(z)$ be a polynomial of degree n for which $|p_n(z)| \leqq 1$ holds as $|z| \leqq 1$; then $|p_n'(z)| \leqq n$ as $|z| \leqq 1$ with $|p_n'(z)| = n$ if and only if $p_n(z) = az^n$, $|a| = 1$.

Some time ago P. Erdös conjectured that *if* $|p_n(z)| \leqq 1$ *holds as* $|z| \leqq 1$ *and* $p_n(z)$ *has no roots inside the unit circle, then* $|p_n'(z)| \leqq n/2$ as $|z| \leqq 1$. In the present note we give a proof of this conjecture.

Preliminaries. Let us introduce the following notation which shall be used throughout this paper:

$$p_n(z) = c \prod_{\nu=1}^{n} (z - z_\nu), \qquad c \neq 0;$$

$$q_n(z) = \bar{c} \prod_{\nu=1}^{n} (1 - z\bar{z}_\nu) = z^n \bar{p}_n(z^{-1}).$$

Then for $|z| = 1$ we have $|p_n(z)| = |q_n(z)|$.

LEMMA I. *If* $p_n(z)$ *has no roots inside the unit circle, that is* $|z_\nu| \geqq 1$, *the polynomial* $p_n(z) + \epsilon q_n(z)$, $|\epsilon| = 1$, *will have all its roots on the unit circle.*[1]

LEMMA II. *If* $p_n(z)$ *has no roots inside the unit circle,* $|z_\nu| \geqq 1$, *we have* $|p_n'(z)| \leqq |q_n'(z)|$ *as* $|z| = 1$.

Let $z \neq z_\nu$; using the abbreviation $z^{-1}z_\nu = A_\nu$ we find

$$|p_n'(z)/p_n(z)| = \left| \sum_{\nu=1}^{n} (z - z_\nu)^{-1} \right| = \left| \sum_{\nu=1}^{n} (1 - A_\nu)^{-1} \right|,$$

$$|q_n'(z)/q_n(z)| = \left| \sum_{\nu=1}^{n} (z - \bar{z}_\nu^{-1})^{-1} \right| = \left| \sum_{\nu=1}^{n} A_\nu(1 - A_\nu)^{-1} \right|.$$

Since $|A_\nu| \geqq 1$, $A_\nu \neq 1$, we obtain

Received by the editors March 8, 1944.

[1] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Berlin, 1925, vol. 1, p. 88, Problem 26.